Assignment 4: Operator Coaster
Due Date: November 4, 2019

1. **Utility of the Unitary (5 Points)**

   Show that if $|i\rangle$'s, where $i = 1, \ldots, N$, form an orthonormal basis, then $\hat{U}|i\rangle$'s, where $\hat{U}$ is a unitary operator, also form a basis.

2. **A Short Drill (15 Points)**

   (a) **(5 Points)** Show that $\mathrm{tr}(\hat{A}\hat{B}) = \mathrm{tr}(\hat{B}\hat{A})$.

   (b) **(5 Points)** Show that the trace of a commutator is always zero.

   (c) **(5 Points)** Show that the trace of a Hermitian operator equals the sum of its eigenvalues.

3. **A Longer One (15 Points)**

   (a) **(5 Points)** A Hermitian operator, $\hat{P}_{HV} = |H\rangle\langle H| - |V\rangle\langle V|$, resolves the polarization state of a photon into $|H\rangle$ and $|V\rangle$. P_{HV} is measured for a beam of photons prepared in the state

   $$|\psi\rangle = \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}}e^{i\frac{\pi}{3}}|V\rangle.$$

   What are the possible outcomes of this measurement? What are their probabilities? For each outcome, what state is the system left in?

   (b) **(5 Points)** In analogy with $\hat{P}_{HV}$, define $\hat{P}_C$, which is the operator corresponding to measurement of circular polarization. Define this operator such that measurements of left-circular polarization yield positive values. Find the matrix representation of $\hat{P}_C$ in the basis $\{|H\rangle, |V\rangle\}$.

   (c) **(5 Points)** A measurement of P_{HV} is performed on a photon prepared in state $|L\rangle$, which represents left-circular polarization. A measurement of P_C (defined in part (a)) is then performed on a second photon prepared in the same state. What is the probability that the first measurement returns $+1$ and the second returns -1?

4. **The Uncertainty (5 Points)**

   Verify that measurements of P_{HV} and P_C satisfy the appropriate indeterminancy relationship for a beam of photons prepared in the state

   $$|\psi\rangle = \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}}e^{i\frac{\pi}{3}}|V\rangle.$$