Assignment 5: From Hops to Pirouettes
Due Date: November 25, 2019

1. **Warming Up (15 Points)**

   Consider the case $j = 1$.

   (a) **(5 Points)** Find the matrices representing the operators $\hat{J}^2$, $\hat{J}_z$, $\hat{J}_\pm$, $\hat{J}_x$, and $\hat{J}_y$. Mention the basis that you use for representation.

   (b) **(5 Points)** Write down the joint eigenstates of $\hat{J}^2$ and $\hat{J}_z$.

   (c) **(5 Points)** Use the matrices of $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ to calculate $[\hat{J}_x, \hat{J}_y]$, $[\hat{J}_y, \hat{J}_z]$, and $[\hat{J}_z, \hat{J}_x]$.

2. **A Pirouetting System (15 Points)**

   Consider a system of total angular momentum $j = 1$. We are interested here in the measurement of $\hat{J}_y$. Its matrix in the simultaneous eigenbasis of $\hat{J}^2$ and $\hat{J}_z$ is given by

$$\hat{J}_y \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

   (a) **(5 Points)** What are the possible measurement results?

   (b) **(10 Points)** Calculate $\langle \hat{J}_z \rangle$, $\langle \hat{J}_z^2 \rangle$, and ΔJ_z if the system is in the state $J_y = \hbar$.

3. **A Handy Tack (10 Points)**

   Consider the operator

$$\hat{A} = \frac{1}{2}(\hat{J}_x \hat{J}_y + \hat{J}_y \hat{J}_x).$$

   Calculate the expectation value of $\hat{A}$ and $\hat{A}^2$ with respect to the state $|j, m\rangle$.

   Hint: Write $\hat{J}_x$ and $\hat{J}_y$ in terms of $\hat{J}_+$ and $\hat{J}_-$.

4. **Another Warm-Up Session (10 Points)**

   Consider the state $|j, m\rangle$.

   (a) **(5 Points)** Show that $\Delta J_x \Delta J_y = \hbar^2[j(j+1) - m^2]/2$, where $\Delta J_x = \sqrt{\langle \hat{J}_x^2 \rangle - \langle \hat{J}_x \rangle^2}$ and the same for ΔJ_y.

(b) **(5 Points)** Show that this relation is consistent with

$$\Delta J_x \Delta J_y \geq (\hbar/2)\left|\left\langle \hat{J}_z \right\rangle\right| = \hbar^2 m/2.$$

5. **A Useful Identity (15 Points)**

Show that

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}.$$

Use this identity to calculate $[\hat{J}^2, \hat{J}_x]$, $[\hat{J}^2, \hat{J}_y]$, and $[\hat{J}^2, \hat{J}_z]$. Explain your results via a physical argument.

Hint: Use $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ as well.

6. **Yet Another Warm-Up (10 Points)**

I promise it would be the last.

(a) **(5 Points)** Consider the following:

$$J_+ \left| j, m \right\rangle = \hbar c_+ \left| j, m+1 \right\rangle.$$

Find c_+ in terms of j and m.

(b) **(5 Points)** Express $\hat{J}_- \hat{J}_+$ in terms of $\hat{J}^2$ and $\hat{J}_z$.

7. **Spin Angular Momentum (25 Points)**

Consider a spin-$\frac{3}{2}$ particle, that is, $j = s = 3/2$.

(a) **(2 Points)** What are the allowed values of the magnetic quantum number, m_s?

(b) **(3 Points)** Consider a simultaneous eigenbasis of $\hat{S}^2$ and $\hat{S}_z$. Call it the Zeeman basis. In a particular representation of the Zeeman basis, the basis states are ordered such that

$$\left| s, s \right\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \left| s, s-1 \right\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and so on.

Suppose that the particle, when expressed in this representation, is in the state

$$|\psi\rangle \doteq N \begin{pmatrix} i \\ 2 \\ 3 \\ 4i \end{pmatrix}.$$

Find N so that $|\psi\rangle$ is normalized.

(c) **(5 Points)** Given that

$$\hat{S}_+|s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s+1)}\ \hbar\, |s, m_s + 1\rangle.$$

Find $\hat{S}_+$ in the chosen representation.

(d) **(5 Points)** Find $\hat{S}_-$ in the chosen representation.

(e) **(5 Points)** Finally, find $\hat{S}_x$ and $\hat{S}_y$ in the chosen representation.

(f) **(5 Points)** Using your answers to part e, verify the following:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z.$$