Quiz 2

A1(a) $\hat{A}|S\rangle = |S\rangle$

$\hat{A}|B\rangle = -|B\rangle$

$\hat{A} = |S\rangle\langle S| - |B\rangle\langle B|$

(b) $\left\{ \underbrace{\frac{1}{\sqrt{2}}|S\rangle + \frac{1}{\sqrt{2}}|B\rangle}_{|\tilde{S}\rangle}, \underbrace{\frac{1}{\sqrt{2}}|S\rangle - \frac{1}{\sqrt{2}}|B\rangle}_{|\tilde{B}\rangle} \right\}$

$|\tilde{S}\rangle + |\tilde{B}\rangle = \frac{2}{\sqrt{2}}|S\rangle$

$|S\rangle = \frac{1}{\sqrt{2}}\left[|\tilde{S}\rangle + |\tilde{B}\rangle\right]$

$|\tilde{S}\rangle - |\tilde{B}\rangle = \frac{2}{\sqrt{2}}|B\rangle$

$|B\rangle = \frac{1}{\sqrt{2}}\left[|\tilde{S}\rangle - |\tilde{B}\rangle\right]$

$\hat{S} = \begin{bmatrix} \langle \tilde{S}|S\rangle & \langle \tilde{S}|B\rangle \\ \langle \tilde{B}|S\rangle & \langle \tilde{B}|B\rangle \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$A = \hat{S}\hat{A}\hat{S}^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} +1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 1/\sqrt{2} & +1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} +1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\hat{A}$ is not diagonal in the new basis, because the new basis states are not

A2. $\hat{A}|a_n\rangle = a_n|a_n\rangle \longrightarrow ①$

$\langle a_k|\hat{A}|a_n\rangle = a_n \langle a_k|a_n\rangle$

Taking the adjoint of ①.

$\langle a_n|A^\dagger = a_n^* \langle a_n|$

But $\hat{A}^\dagger = -\hat{A}$ (anti Hermitian)

$-\langle a_n|\hat{A} = a_n^* \langle a_n|$

$-\langle a_n|\hat{A}|a_n\rangle = a_n^* \langle a_n|a_n\rangle$

$-a_n = a_n^*$

So, a_n is purely imaginary.

Hermitian operators are for observables so their eigenvalues must be real.

A3 lets consider the eigenstate $|z\rangle$ i.e.

$\hat{S}_z|z\rangle = \frac{\hbar}{2}|z\rangle$

$\langle \hat{S}_x\rangle = \langle z|\hat{S}_x|z\rangle$

$= \frac{\hbar}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$

$= 0$

$\langle S_x^2\rangle = \langle z|\hat{S}_x^2|z\rangle$

$= \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$

$= \frac{\hbar^2}{4} 1 = \frac{\hbar^2}{4}$

$\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}$

A4. In the basis $\{|1\rangle, |2\rangle, |3\rangle\}$ we have.

$$\hat{A} = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One eigenstate is clearly $|3\rangle$

For the other two, look at the upper 2×2 block

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \longrightarrow \sigma_y$$

Eigenstates of σ_y = $\frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)$, $\frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle)$

eigenvalues.

$$\begin{pmatrix} -\lambda & i \\ i & -\lambda \end{pmatrix}$$

$\therefore \lambda^2 - 1 = 0$

$\lambda = \pm 1$.

eigenvalue.	1	eigenstate:	$\frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle)$
	-1		$\frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)$
	1		$|3\rangle$

$$\hat{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$