# Assignment #4
(Solution)

(1) $(\hat{u}|i\rangle)^\dagger(\hat{u}|j\rangle)$ $(i = 1, \dots, N$ and $j = 1, \dots, N)$

$= \langle i|\hat{u}^\dagger \hat{u}|j\rangle$

$= \langle i|\hat{u}^{-1}\hat{u}|j\rangle$ $(\because \hat{u}$ is unitary$)$

$= \langle i|\hat{1}|j\rangle$

$= \langle i|j\rangle$

$= \delta_{ij}.$ $(\because |i\rangle$'s are orthonormal$)$

Clearly, $\hat{u}|i\rangle$'s are orthonormal.

Now,

$\sum_i \hat{u}|i\rangle\langle i|\hat{u}^\dagger$

$= \hat{u}\left(\sum_i |i\rangle\langle i|\right)\hat{u}^\dagger$

$= \hat{u}\,\hat{1}\,\hat{u}^\dagger$ $(\because |i\rangle$'s form a (complete) basis$)$

$= \hat{u}\hat{u}^{-1}$ $(\because \hat{u}$ is unitary$)$

$= \hat{1}.$

Clearly, $\hat{u}|i\rangle$'s satisfy the completeness relation.

Orthonormality and completeness imply that $\hat{u}|i\rangle$'s form an orthonormal basis. Q.E.D.

(2) (a) $tr(\hat{A}\hat{B}) = \sum_i \langle i|(\hat{A}\hat{B})|i\rangle$

$$= \sum_i \sum_j \langle i|\hat{A}|j\rangle\langle j|\hat{B}|i\rangle$$

$$= \sum_j \sum_i \langle j|\hat{B}|i\rangle\langle i|\hat{A}|j\rangle$$

$$= \sum_j \langle j|\hat{B}\,\hat{1}\,\hat{A}|j\rangle$$

$$= \sum_j \langle j|(\hat{B}\hat{A})|j\rangle = tr(\hat{B}\hat{A}). \quad \text{Q.E.D.}$$

(b) Consider the commutator $[\hat{A}, \hat{B}]$.

$$tr([\hat{A}, \hat{B}])$$

$$= tr(\hat{A}\hat{B} - \hat{B}\hat{A})$$

$$= tr(\hat{A}\hat{B}) - tr(\hat{B}\hat{A})$$

$$= tr(\hat{A}\hat{B}) - tr(\hat{A}\hat{B}) \quad (\because tr(\hat{A}\hat{B}) = tr(\hat{B}\hat{A}))$$

$$= 0. \quad \text{Q.E.D.}$$

(c) Considering the eigenbasis of $\hat{A}$, a Hermitian operator, we have the following:

$$tr(\hat{A}) = \sum_i \langle i|\hat{A}|i\rangle$$

$$= \sum_i \langle i|\lambda_i|i\rangle$$

$$= \sum_i \lambda_i \langle i|i\rangle$$

$$= \sum_i \lambda_i.$$

Considering an arbitrary basis, we have the following:

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$\mathrm{tr}(\hat{A})$

$$= \sum_j \langle \tilde{j} | \hat{A} | \tilde{j} \rangle$$

$$= \sum_j \sum_i \sum_k \langle \tilde{j} | k \rangle \langle k | \hat{A} | i \rangle \langle i | \tilde{j} \rangle$$

$$= \sum_i \sum_k \sum_j \langle i | \tilde{j} \rangle \langle \tilde{j} | k \rangle \langle k | \hat{A} | i \rangle$$

$$= \sum_i \sum_k \delta_{ik} \langle k | \hat{A} | i \rangle$$

$$= \sum_i \langle i | \hat{A} | i \rangle$$

$$= \sum_i \langle i | \lambda_i | i \rangle$$

$$= \sum_i \lambda_i \langle i | i \rangle$$

$$= \sum_i \lambda_i. \quad \text{Q.E.D.}$$

(3) (a) $\hat{P}_{HV} = |H\rangle\langle H| - |V\rangle\langle V|.$

Represented in the basis $\{|H\rangle, |V\rangle\}$, $\hat{P}_{HV}$ is diagonal, something implying that $|H\rangle$ and $|V\rangle$ form an eigenbasis and that the corresponding eigenvalues are $+1$ and $-1$.

Possible measurement outcomes are $+1$ and $-1$, and their respective probabilities are the following:

$$P(+1) = \| \langle H | \psi \rangle \|^2$$

$$= \left\| \frac{1}{\sqrt{3}} \right\|^2$$

$$= \frac{1}{3}.$$

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$$P(-1) = \| \langle V | \psi \rangle \|^2$$

$$= \left\| \sqrt{\frac{2}{3}}\, e^{i\frac{\pi}{3}} \right\|^2$$

$$= \frac{2}{3}.$$

If the outcome is +1, the system is left in the state $|H\rangle$.
If the outcome is −1, the system is left in the state $|V\rangle$.

(b) $$\hat{P}_c = |L\rangle\langle L| - |R\rangle\langle R|$$

$$\hat{P}_c = \left(\frac{1}{\sqrt{2}}|H\rangle + \frac{i}{\sqrt{2}}|V\rangle\right)\left(\frac{1}{\sqrt{2}}\langle H| - \frac{i}{\sqrt{2}}\langle V|\right)$$
$$- \left(\frac{1}{\sqrt{2}}|H\rangle - \frac{i}{\sqrt{2}}|V\rangle\right)\left(\frac{1}{\sqrt{2}}\langle H| + \frac{i}{\sqrt{2}}\langle V|\right)$$

$$\hat{P}_c = \frac{1}{2}\left(|H\rangle\langle H| - i|H\rangle\langle V| + i|V\rangle\langle H| + |V\rangle\langle V|\right)$$
$$- \frac{1}{2}\left(|H\rangle\langle H| + i|H\rangle\langle V| - i|V\rangle\langle H| + |V\rangle\langle V|\right)$$

$$\hat{P}_c = -i|H\rangle\langle V| + i|V\rangle\langle H|$$

$$\hat{P}_c \doteq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

(c) $$P_1(+1) = \| \langle H | L \rangle \|^2$$

$$P_1(+1) = \left\| \langle H | \left(\frac{1}{\sqrt{2}}|H\rangle + \frac{i}{\sqrt{2}}|V\rangle\right) \right\|^2 = \frac{1}{2}.$$

$$P_2(-1) = \| \langle R | L \rangle \|^2$$

$$P_2(-1) = 0.$$

Measurements are independent; therefore,

$$P_{12}(+1, -1) = P_1(+1) \times P_2(-1)$$

$$= \frac{1}{2} \times 0 = 0.$$

(4) $$\Delta P_{HV} = \sqrt{\langle \hat{P}_{HV}^2 \rangle_\psi - \langle \hat{P}_{HV} \rangle_\psi^2}.$$

$$\langle \hat{P}_{HV}^2 \rangle_\psi = \left( \frac{1}{\sqrt{3}} \langle H| + \sqrt{\frac{2}{3}} \langle V| e^{-i\frac{\pi}{3}} \right) \hat{P}_{HV} \hat{P}_{HV} \left( \frac{1}{\sqrt{3}} |H\rangle + \sqrt{\frac{2}{3}} |V\rangle e^{i\frac{\pi}{3}} \right)$$

$$= \left( \frac{1}{\sqrt{3}} \langle H| + \sqrt{\frac{2}{3}} \langle V| e^{-i\frac{\pi}{3}} \right) \hat{P}_{HV} \left( \frac{1}{\sqrt{3}} |H\rangle - \sqrt{\frac{2}{3}} |V\rangle e^{i\frac{\pi}{3}} \right)$$

$$= \left( \frac{1}{\sqrt{3}} \langle H| + \sqrt{\frac{2}{3}} \langle V| e^{-i\frac{\pi}{3}} \right) \left( \frac{1}{\sqrt{3}} |H\rangle + \sqrt{\frac{2}{3}} |V\rangle e^{i\frac{\pi}{3}} \right)$$

$$= 1.$$

$$\langle \hat{P}_{HV} \rangle_\psi = \left( \frac{1}{\sqrt{3}} \langle H| + \sqrt{\frac{2}{3}} \langle V| e^{-i\frac{\pi}{3}} \right) \hat{P}_{HV} \left( \frac{1}{\sqrt{3}} |H\rangle + \sqrt{\frac{2}{3}} |V\rangle e^{i\frac{\pi}{3}} \right)$$

$$= \left( \frac{1}{\sqrt{3}} \langle H| + \sqrt{\frac{2}{3}} \langle V| e^{-i\frac{\pi}{3}} \right) \left( \frac{1}{\sqrt{3}} |H\rangle - \sqrt{\frac{2}{3}} |V\rangle e^{i\frac{\pi}{3}} \right)$$

$$= \frac{1}{3} - \frac{2}{3} = \frac{-1}{3}$$

$$\Delta P_{HV} = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}}.$$

$$\Delta P_C = \sqrt{\langle \hat{P}_C^2 \rangle_\psi - \langle \hat{P}_C \rangle_\psi^2}.$$

$$\langle \hat{P}_C^2 \rangle_\psi \doteq \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} e^{-i\frac{\pi}{3}} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} e^{i\frac{\pi}{3}} \end{bmatrix}$$

$$\langle \hat{P}_c^2 \rangle_\psi = 1.$$

$$\langle \hat{P}_c \rangle_\psi = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\, e^{-i\frac{\pi}{3}} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}}\, e^{i\frac{\pi}{3}} \end{bmatrix}$$

$$\langle \hat{P}_c \rangle_\psi = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\, e^{-i\frac{\pi}{3}} \end{bmatrix} \begin{bmatrix} -i\sqrt{\frac{2}{3}}\, e^{i\frac{\pi}{3}} \\ i\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\langle \hat{P}_c \rangle_\psi = -i\frac{\sqrt{2}}{3} e^{i\frac{\pi}{3}} + i\frac{\sqrt{2}}{3} e^{-i\frac{\pi}{3}}$$

$$\langle \hat{P}_c \rangle_\psi = -i\frac{\sqrt{2}}{3}\left(e^{i\frac{\pi}{3}} - e^{-i\frac{\pi}{3}}\right) = 2\frac{\sqrt{2}}{3}\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{2\sqrt{2}}{3}\,\frac{\sqrt{3}}{2}$$

$$= \sqrt{\frac{2}{3}}.$$

$$\Delta P_c = \sqrt{1 - \left(\sqrt{\frac{2}{3}}\right)^2} = \sqrt{\frac{1}{3}}.$$

Now,

$$[\hat{P}_{HV}, \hat{P}_c] \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} - \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2i \\ -2i & 0 \end{bmatrix}.$$

$$\langle [\hat{P}_{HV}, \hat{P}_c] \rangle_\psi = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\, e^{-i\frac{\pi}{3}} \end{bmatrix} \begin{bmatrix} 0 & -2i \\ -2i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}}\, e^{i\frac{\pi}{3}} \end{bmatrix}$$

$$= -2i \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\, e^{-i\frac{\pi}{3}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{2}{3}}\, e^{i\frac{\pi}{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= -2i\left(\frac{\sqrt{2}}{3}e^{i\frac{\pi}{3}} + \frac{\sqrt{2}}{3}e^{-i\frac{\pi}{3}}\right)$$

$$= -2i\,\frac{\sqrt{2}}{3}\left(2\cos\left(\frac{\pi}{3}\right)\right)$$

$$= -\frac{2\sqrt{2}}{3}i.$$

Now, checking the uncertainty relation, we have the following:

$$\Delta P_{HV}\,\Delta P_c \overset{?}{\geq} \frac{1}{2}\left\| \langle [\hat{P}_{HV}, \hat{P}_c] \rangle_{\psi} \right\|$$

$$\sqrt{\frac{8}{9}}\,\sqrt{\frac{2}{3}} \overset{?}{\geq} \frac{1}{2}\,\frac{2\sqrt{2}}{3}$$

$$\frac{2\sqrt{2}}{3} \times \frac{\sqrt{2}}{\sqrt{3}} \overset{?}{\geq} \frac{\sqrt{2}}{3}$$

$$\frac{4}{3\sqrt{3}} \overset{?}{\geq} \frac{\sqrt{2}}{3}$$

$$\frac{\sqrt{8}}{\sqrt{3}} \overset{?}{\geq} 1$$

$$\sqrt{\frac{8}{3}} \overset{\checkmark}{\geq} 1. \quad \text{Q.E.D.}$$