Solution Mid-Term
PHY 212: Quantum Mechanics I

Formula Sheet:

1. Pauli matrices are: $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
2. A useful theorem is: $\exp(-i\theta\hat{A}) = \cos\theta\hat{1} - i\sin\theta\hat{A}$ if $\hat{A}^2 = \hat{1}$.
3. Angular momentum operators for spin 1/2 system are:
   $\hat{S}_x = \frac{\hbar}{2}\,\hat{\sigma}_x, \quad \hat{S}_y = \frac{\hbar}{2}\,\hat{\sigma}_y, \quad \hat{S}_z = \frac{\hbar}{2}\,\hat{\sigma}_z.$
4. A useful identity is: $\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{\sigma}_z^2 = \hat{1}$.

Attempt all questions.

1. If $\hat{A}$ and $\hat{B}$ are Hermitian, show that $\hat{C} = [\hat{A}, \hat{B}]$ is anti- Hermitian. [5 **marks**]

   Answer 1

$$
\begin{aligned}
[\hat{A}, \breve{B}]^\dagger &= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\
&= (\hat{A}\hat{B})^\dagger - (\hat{B}\hat{A})^\dagger \\
&= \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger \\
&= \hat{B}\hat{A} - \hat{A}\hat{B} \\
&= [\hat{B}, \breve{A}] \\
&= -[\hat{A}, \breve{B}]
\end{aligned}
$$

   So $[\hat{A}, \breve{B}]$ is anti- Hermitian.

2. If $|a_n\rangle\,'s$ form an orthogonal basis for a Hilbert space, show that $\hat{U}\,|a_n\rangle\,'s$ also form a basis if $\hat{U}$ is a unitary operator. [5 **marks**]

   Answer 2

   Let $|a_n\rangle = |\phi_n\rangle$. The $|a_n\rangle\,'s$ will form a basis if

$$\sum_n |\phi_n\rangle\langle\phi_n| = \hat{1}$$

Let's check this.

$$\begin{aligned}\sum_n |\phi_n\rangle\langle\phi_n| &= \sum_n \hat{U}|a_n\rangle\langle a_n|\hat{U}^\dagger \\ &= \hat{U}\left(\sum_n |a_n\rangle\langle a_n|\right)\hat{U}^\dagger \\ &= \hat{U}\hat{U}^\dagger \\ &= \hat{U}\hat{U}^{-1} \\ &= \hat{1}\end{aligned}$$

3. (a) $\hat{A}$ is a Hermitian operator. Let $\{|1\rangle, |2\rangle, ...|N\rangle\}$ form an orthonormal basis for the Hilbert space containing an arbitrary state $|\psi\rangle$. Show that

$$\langle\hat{A}^2\rangle = \sum_{i=1}^{N} \left|\langle\psi|\hat{A}|i\rangle\right|^2.$$ [5 **marks**]

(b) If $\{|\lambda_1\rangle, |\lambda_2\rangle, ...|\lambda_N\rangle\}$ are nondegenerate eigenstates of the Hermitian operator $\hat{A}$ with eigenvalues $\{\lambda_1, \lambda_2, ...\lambda_N\}$, show that for an arbitrary state $|\psi\rangle$,

$$\langle\hat{A}\rangle = \sum_{i=1} \lambda_i \left|\langle\psi|\lambda_i\rangle\right|^2.$$ [5 **marks**]

Answer 3

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(a) The expectation value of $\hat{A}^2$ can be found as

$$\begin{aligned}\langle\hat{A}^2\rangle &= \langle\psi|\hat{A}\hat{A}|\psi\rangle \\ &= \langle\psi|\hat{A}\mathbf{\hat{1}}\hat{A}|\psi\rangle \\ &= \langle\psi|\hat{A}\left(\sum_i |i\rangle\langle i|\right)\hat{A}|\psi\rangle \\ &= \sum_i \langle\psi|\hat{A}|i\rangle\langle i|\hat{A}|\psi\rangle \\ &= \sum_i \langle i|\hat{A}^\dagger|\psi\rangle\langle\psi|\hat{A}|i\rangle \qquad \because \hat{A} \text{ is Hermitian} \\ &= \sum_i \left|\langle\psi|\hat{A}|i\rangle\right|^2,\end{aligned}$$

is the required relation.

(b) Similarly, following the same procedure as we did in part (a)

$$\begin{aligned}\langle\hat{A}\rangle &= \langle\psi|\hat{A}|\psi\rangle \\ &= \langle\psi|\hat{A}\hat{\mathbf{1}}|\psi\rangle \\ &= \langle\psi|\hat{A}\Big(\sum_i |\lambda_i\rangle\langle\lambda_i|\Big)|\psi\rangle \\ &= \sum_i \langle\psi|\hat{A}|\lambda_i\rangle\langle\lambda_i|\psi\rangle \\ &= \sum_i \lambda_i\langle\psi|\lambda_i\rangle\langle\lambda_i|\psi\rangle \\ &= \sum_{i=1}^{N} \lambda_i\big|\langle\psi|\lambda_i\rangle\big|^2.\end{aligned}$$

4. A spin 1/2 system is in the state $|+z\rangle$ where $|\pm z\rangle$ are eigenstates of the $\hat{S}_z$ spin operator. An experimentalist intends to apply a rotation operator $e^{-i\pi\hat{S}_y/\hbar}$, desiring to convert the state $|+z\rangle$ into $|-z\rangle$.

   (a) Show that $e^{-i\pi\hat{S}_y/\hbar}$ indeed does the job. [5 **marks**]

   (b) Show that the inner product between the final state and the desired state $|-z\rangle$ is 1. We will use the square of this inner product is a figure of merit for the inversion operation. [5 **marks**]

   (c) Now comes the practical implementation of this rotation! It appears that applying a rotation about the $\hat{S}_y$ axis in the lab is prone to errors. The instrument's y axis is mis-calibrated: when the experimenter applies a pulse nominally rotating the state through θ about the $\hat{S}_y$ axis, she is in fact applying $\theta(1+\varepsilon)$ rotation where ε is a small number ($|\varepsilon| \ll 1$). What is the final state if this erroneous inversion pulse is applied (with a nominal rotation of π radians)? [7 **marks**]

   (d) What is the figure of merit for this erroneous pulse? [3 **marks**]

Answer 4

(a) The candidate rotation operator is

$$\begin{aligned}\hat{R}_y(\pi) &= e^{-i\pi\hat{S}_y/\hbar} = e^{-i\left(\frac{\pi}{2}\right)\left(\frac{2\hat{S}_y}{\hbar}\right)} \\ &= \cos\Big(\frac{\pi}{2}\Big)\hat{\mathbf{1}} - i\sin\Big(\frac{\pi}{2}\Big)\hat{\sigma_y} = -i\hat{\sigma_y}.\end{aligned}$$

Its action on $|z\rangle$ yields

$$\begin{aligned}\hat{R}_y(\pi)\,|z\rangle &= -i\hat{\sigma_y}\,|z\rangle \\ \text{Now}\ \ \hat{\sigma_y}\,|z\rangle &= \Big(-i\,|z\rangle\,\langle -z| \ + \ i\,|-z\rangle\,\langle z|\Big)|z\rangle = i\,|-z\rangle \\ \therefore \hat{R}_y(\pi)\,|z\rangle &= -i(i\,|-z\rangle) = |-z\rangle\,,\end{aligned}$$

which is the desired state. So the rotation operator indeed does the job.

(b) The inner product between the final state $|\psi_{\text{final}}\rangle$ and $|\psi_{\text{des}}\rangle$ is

$$\langle\psi_{\text{des}}\,|\psi_{\text{final}}\rangle = \langle -z|-z\rangle = 1.$$

The figure of merit is $\left|\langle\psi_{\text{des}}\,|\psi_{\text{final}}\rangle\right|^2 = 1$, the maximum, one.

(c) If the experimenter is rotating the state $|z\rangle$ through angle $\pi(1+\varepsilon)$ about the $\hat{S}_y$ axis instead of π, then the final state is

$$|\psi'_{\text{final}}\rangle \;=\; e^{-i\pi(1+\varepsilon)\hat{S}_y/\hbar}\,|z\rangle\,.$$

The erroneous rotation operator is

$$\begin{aligned}e^{-i\pi(1+\varepsilon)\hat{S}_y/\hbar} &= e^{-i\left(\frac{\pi}{2}(1+\varepsilon)\right)\left(\frac{2\hat{S}_y}{\hbar}\right)} \\ &= e^{-i\left(\frac{\pi}{2}(1+\varepsilon)\right)\hat{\sigma_y}} \\ &= \cos\left(\frac{\pi}{2}(1+\varepsilon)\right)\hat{\mathbf{1}} - i\sin\left(\frac{\pi}{2}(1+\varepsilon)\right)\hat{\sigma_y}.\end{aligned}$$

So the final state is

$$\begin{aligned}|\psi'_{\text{final}}\rangle &= \left[\cos\left(\frac{\pi}{2}(1+\varepsilon)\right)\hat{\mathbf{1}} - i\sin\left(\frac{\pi}{2}(1+\varepsilon)\right)\hat{\sigma_y}\right]|z\rangle \\ &= \cos\left(\frac{\pi}{2}(1+\varepsilon)\right)|z\rangle + \sin\left(\frac{\pi}{2}(1+\varepsilon)\right)|-z\rangle\,. \qquad \because \hat{\sigma_y}\,|z\rangle = i\,|-z\rangle\end{aligned}$$

(d) The inner product between the final state $|\psi'_{\text{final}}\rangle$ and $|\psi_{\text{des}}\rangle$ is

$$\begin{aligned}\langle\psi_{\text{des}}|\psi'_{\text{final}}\rangle &= \langle -z|\left[\cos\left(\frac{\pi}{2}(1+\varepsilon)\right)|z\rangle + \sin\left(\frac{\pi}{2}(1+\varepsilon)\right)|-z\rangle\right] \\ &= \sin\left(\frac{\pi}{2}(1+\varepsilon)\right),\end{aligned}$$

and the figure of merit is,

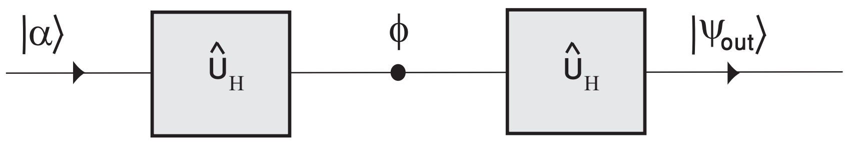
$$\begin{aligned}\text{FOM} &= \sin^2\left(\frac{\pi}{2}(1+\varepsilon)\right)\\ &= \frac{1-\cos\pi(1+\varepsilon)}{2}.\end{aligned}$$

Now

$$\begin{aligned}\cos\pi(1+\varepsilon) &= \cos\pi\cos\pi\varepsilon - \sin\pi\sin\pi\varepsilon\\ &= -\cos\pi\varepsilon\\ &= -\left(1-\frac{(\pi\varepsilon)^2}{2!}+\frac{(\pi\varepsilon)^4}{4!}-\frac{(\pi\varepsilon)^6}{6!}+\cdots\cdot\right)\\ \therefore \text{FOM} &= \frac{1}{2}-\frac{1}{2}\cos\pi(1+\varepsilon)\\ &= \frac{1}{2}+\frac{1}{2}\left(1-\frac{(\pi\varepsilon)^2}{2!}+\frac{(\pi\varepsilon)^4}{4!}-\frac{(\pi\varepsilon)^6}{6!}+\cdots\cdot\right).\end{aligned}$$

Since $\varepsilon \ll 1$, the higher order terms will be progressively smaller. However the presence of ε makes the FOM smaller than one.

5. A two-dimensional Hilbert space is spanned by the basis states $|\alpha\rangle$ and $|\beta\rangle$. Three quantum logic gates act on a state initialized as $|\alpha\rangle$ (a) first a Hadamard gate, (b) then a phase gate and (c) finally another Hadamard gate. The quantum circuit is shown here.



The action of the gates on the basis states is given by these prescriptions.

$$\begin{aligned}\hat{U}_H|\alpha\rangle &= \frac{1}{\sqrt{2}}\big(|\alpha\rangle+|\beta\rangle\big)\\ \hat{U}_H|\beta\rangle &= \frac{1}{\sqrt{2}}\big(|\alpha\rangle-|\beta\rangle\big)\\ \hat{U}_\phi|\alpha\rangle &= |\alpha\rangle\\ \hat{U}_\phi|\beta\rangle &= e^{i\phi}|\beta\rangle.\end{aligned}$$

Find the output state $|\psi_{\text{out}}\rangle$. What is the probability that upon measurement in the $\{|\alpha\rangle, |\beta\rangle\}$ basis, the output state is $|\beta\rangle$? [10 **marks**]

Answer 5

When the first Hadamard gate acts on initial state $|\alpha\rangle$, the output state is

$$\hat{U}_H|\alpha\rangle = |\psi_1\rangle = \frac{1}{\sqrt{2}}\big(|\alpha\rangle + |\beta\rangle\big),$$

then a phase gate operates on this state $|\psi_1\rangle$,

$$\begin{aligned}\hat{U}_\phi\,|\psi_1\rangle &= \hat{U}_\phi\left(\frac{1}{\sqrt{2}}\big(|\alpha\rangle + |\beta\rangle\big)\right)\\ &= \frac{1}{\sqrt{2}}\big(|\alpha\rangle + e^{i\phi}\,|\beta\rangle\big).\end{aligned}$$

Finally, another Hadamard gate acts on this state,

$$\begin{aligned}|\psi_{\text{out}}\rangle &= \hat{U}_H\left(\frac{1}{\sqrt{2}}\big(|\alpha\rangle + e^{i\phi}\,|\beta\rangle\big)\right)\\ &= \frac{1}{\sqrt{2}}\left(\hat{U}_H\,|\alpha\rangle + e^{i\phi}\hat{U}_H\,|\beta\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\big(|\alpha\rangle + |\beta\rangle\big) + e^{i\phi}\frac{1}{\sqrt{2}}\big(|\alpha\rangle - |\beta\rangle\big)\right)\\ &= \frac{1}{2}\left((1+e^{i\phi})\,|\alpha\rangle + (1-e^{i\phi})\,|\beta\rangle\right)\end{aligned}$$

The probability for preparing the state $|\beta\rangle$ is

$$\begin{aligned}P\left(|\beta\rangle\,\middle|\,|\psi_{\text{out}}\rangle\right) &= \left|\langle\beta\,|\psi_{\text{out}}\rangle\right|^2\\ \text{Now}\ \ \langle\beta\,|\psi_{\text{out}}\rangle &= \langle\beta|\left(\frac{1}{2}(1+e^{i\phi})\,|\alpha\rangle + \frac{1}{2}(1-e^{i\phi})\,|\beta\rangle\right)\\ &= \frac{1}{2}(1-e^{i\phi})\\ \therefore \left|\langle\beta\,|\psi_{\text{out}}\rangle\right|^2 &= \left(\frac{1}{2}(1-e^{-i\phi})\right)\left(\frac{1}{2}(1-e^{i\phi})\right)\\ &= \frac{1}{4}\big(1-e^{i\phi}-e^{-i\phi}+1\big)\\ &= \frac{2}{4}\left(1-\frac{e^{i\phi}+e^{-i\phi}}{2}\right)\\ &= \frac{1}{2}(1-\cos\phi) = \sin^2\big(\frac{\phi}{2}\big).\end{aligned}$$

We have achieved a quantum interferometer where the probability of receiving a photon on the β channel is a function of ϕ. This interferometer is also a simple quantum computer.