The Magnetic Pendulum

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Nonlinearity is a profound concept in the study of physical systems. The characteristics of seemingly very simple systems may turn out to be extremely intricate due to non-linearity. The study of chaos also begins with the study of such simple systems. The magnetic pendulum is one such system.

A pendulum is one of the simplest and diverse systems in terms of its mathematical basis and the range of fields of science that it can relate to. With slight modifications, it can exhibit exotic, mathematically rich phenomena. In this experiment, we will explore the notion of nonlinear and chaotic dynamics using a “magnetic pendulum”.

**KEYWORDS** Determinism · Chaos · Supersensitivity · Phase Portrait · Poincare Map · Attractor · Resonance.

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1 Objectives

In this experiment, we will discover:

1. how apparently simple systems can be highly non-linear and exhibit a complex behavior under certain conditions,

2. the physical structure of dynamical systems, and

3. the conditions and consequences of the notion of super-sensitivity and its relationship with chaos through simulation and experiment.

References


2 Apparatus

The apparatus comprises a pendulum with adjustable height and detachable permanent disc magnets. The pendulum itself is attached to a crank mechanism that is placed horizontally. The amplitude of the crank can be switched between three values by manually changing the crank pin. The stepper motor that drives the crank mechanism is attached to an Arduino based controller which can connect to the MagPendulum Software (magpendulum.exe) provided with the apparatus, through a USB cable. The velocity and the initial phase of the stepper motor can easily be controlled using the software.

![Apparatus Image]

Figure 1: A photograph showing various components of Physlab’s magnetic pendulum. Parts are in metal and plastic, the latter being 3D printed.
Another disc magnet can be placed underneath the pendulum using the magnet holder. The height and normal position of this holder can also be changed using the knobs screwed with the holder.

Angular displacement of the pendulum is measured by a homemade optical angle sensor attached with the hinge of the pendulum. These values can be acquired in the MagPendulum Software using a couple of data presentation tools. Data can be saved and recorded. We use the term save for past data and record for future data. The graphs can also be cleared and the axes can be rescaled at will. A few embellishments such as changing the color of the trajectories and time series graphs are also available in the software.

3 The Experiment

The current experiment is divided into three parts.

1. Exploring nonlineairities in a simple pendulum through simulation and experiment;

2. acquiring time orbits and phase space diagrams of nonlinear systems, also through simulation and experiment.

3.1 Exploring Non-Linearities

3.1.1 Simulation

We start off this experiment through a pre-lab exercise exploring nonlinearity. Of course, the purpose is to see how a simple pendulum can become nonlinear. In this regime, the assertion that the time period is independent of the initial pendulum breaks down. If we ignore friction a simple pendulum’s equation of motion can be written as:

$$\ddot{\theta} + \frac{g}{l} \sin(\theta) = 0.$$  \hspace{1cm} (1)

Now if we make the small angle approximation i.e. \( \sin(\theta) \approx \theta \) the we obtain:

$$\ddot{\theta} + \frac{g}{l} \theta = 0$$  \hspace{1cm} (2)
Here, \( \theta \) is the angular displacement of the pendulum of the length, \( L \), and \( g \) is the acceleration due to gravity. In this manual, dots over the variable represent time derivatives. For example \( \dot{\theta} \) represents the angular velocity \( \omega \) and \( \ddot{\theta} = \dot{\omega} \) represents the angular acceleration.

**Question** What factors control the time period of the pendulum in the case of small angles? Does time period depend on the initial amplitude from which the pendulum is displaced? Derive Eq. (1) from Newton’s force equation.

For the large amplitude, \( \theta_o \), where the small angle approximation breaks down, we can re-arrange and integrate Equation (2) to give us the following formula for time period of a pendulum:

\[
T(\theta_o) = 4 \sqrt{\frac{l}{g}} K(k) \quad (3)
\]

where,

\[
K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} \quad (4)
\]

is called the complete elliptic integral of the first kind and can be easily tabulated through Matlab or looked up in standard references. Here \( k = \sin \frac{\theta_o}{2} \). For example, in Matlab\(^1\), this integral can be numerically computed using the function `EllipticK`. Further details about this integral and its usage to determine the time period of a nonlinear pendulum can be seen in the reference [4].

**Exercise** Plot a graph time period versus \( \theta_o \). Vary values of \( \theta_o \) from \( 0^\circ \) to \( 90^\circ \). Choose an appropriate value of the length of the pendulum (that matches your apparatus). At what point could you say the transition from linearity to non-linearity occurs?

**Question** With the help of the reference [4], derive Eq. (4).

### 3.1.2 Experiment

Let’s now, in an experiment, observe this transition from linearity to non-linearity. Double click the program `mappendulum.exe` and acquire the data. Note down the time period for

\(^1\) These commands require Matlab’s Symbolic Math Toolbox. Type `ver` in your Matlab command window to check and see if you have this toolbox.
varying $\theta_o$.

**Exercise** Do your experimental observations agree with the simulation?

### 3.2 Temporal trajectories and phase space portraits

#### 3.2.1 Modeling the magnetic pendulum

In this part of the experiment you will steadily decrease the distance, $d$, between the two magnets and observe the changes that occur in the time course plots and the phase portraits of the pendulum. This is a two dimensional system with the canonical coordinates being $\theta$ and $\omega \dot{\theta}$. A phase portrait gives the phase space picture, with $\dot{\theta}$ plotted against $\theta$ as time progresses. We are required to observe and interpret the time series diagrams and the phase portraits. These figures are interactively built up by the computer program and the data can also be recorded for offline analysis. We will see how the transition from linear to nonlinear behavior occurs, whether the period doubles and the we will also look for the onset of chaotic behavior.

**Sketch** Draw the phase portrait of a simple pendulum in its linear regime.

We will first gain insight by simulating the motion of our pendulum in Matlab. For this purpose, we will need to know the equation of motion of the pendulum. This will then be numerically solved. This, in turn, necessitates the need for a ‘model’ of the system. The motion exhibited by our pendulum is of the rotational kind. Hence knowing all the torques $\tau$ will allow us to write its equation of motion. Figure 2 identifies the various torques acting on the pendulum. Newton’s Second Law gives us:

$$l \frac{d^2 \theta}{dt^2} = \Sigma \tau_i$$  \hspace{1cm} (5)

where $l$ is the moment of inertia of the pendulum and $\Sigma \tau_i$ is the vector sum of all the torques acting on the pendulum:

$$\Sigma \tau_i = \tau_{\text{gravity}} + \tau_{\text{driver}} + \tau_{\text{damping}} + \tau_{\text{magnetic}}$$  \hspace{1cm} (6)

The torques in this equation are self-explanatory. Substituting the expressions for each of these torques yields the following differential equations [2]:

6
Figure 2: Schematic diagram of the physical pendulum and the forces acting on it. The distance \( d \) is between magnets when the pendulum is in the resting position. The arrows show the magnetic moment vectors. The angle \( \theta \) is measured with an angle sensor. The angle is numerically differentiated to estimate the angular velocity.

\[
\dot{\theta} = \omega \\
\frac{ML^2}{3} \dot{\omega} = -\frac{L}{2} M g \sin \theta + T_{\text{driver}} \sin(\Omega t) - \gamma \omega \\
+ \frac{|\theta| L \mu_0 \frac{m_1 m_2}{4 \pi r^2}}{4 \pi} \cos \left( |\theta| + \tan^{-1} \left( -\frac{h}{L \sin \theta} \right) \right).
\]

(7)  

(8)

The left hand side is simply a manifestation of the L.H.S. of Eq. 5, \( M \) being the mass and \( L \) being the length of the pendulum. The second term on the right side is the torque produced by the force exerted by the reciprocating crankshaft, \( T_{\text{driver}} \) representing the maximum torque and \( \Omega \) being the frequency of the reciprocatory motion. The third term on the R.H.S. is a damping term which is proportional to the angular velocity \( \omega \) and \( \gamma \) is a damping coefficient. The fourth term arises due to a magnetic dipolar attraction between the two magnets, which are each assumed to be magnetic dipoles of strengths \( m_1 \) and \( m_2 \). The vectorial magnetic force can be approximated as:
\[
\mathbf{F}_{\text{magnetic}} = \frac{\mu_0 m_1 m_2 \hat{r}}{4\pi r_\theta^2}. \tag{9}
\]

Equation (9) assumes a Coulomb-like inverse square law between two magnetic moments \( m_1 \) and \( m_2 \) separated by a distance \( r_\theta \). The permeability of vacuum is \( \mu_0 \). The unit vector \( \hat{r} \) points radially away from the line joining the two magnets and indicates the direction of the magnetic force. Furthermore, \( r_\theta = \sqrt{(L \sin \theta)^2 + h_\theta^2} \) and \( h_\theta = d + L(1 - \cos \theta) \). The variables \( d \), \( r_\theta \) and \( h_\theta \) are also shown in Figure 2.

**Exercise** Interpret each term of Equation (8).

### 3.2.2 Simulating the magnetic pendulum

We now aim at numerically solving this equation. This is called 'simulation'. For this purpose, Eq. (8) can be simplified and rewritten as:

\[
\dot{\omega} = -A \sin \theta + B \sin(\Omega t) - C \omega + \frac{|\theta| E}{\theta} \frac{E}{r_\theta^2} \times \cos \left( |\theta| + \tan^{-1} \left( \frac{|\theta|}{\frac{h_\theta}{L \sin \theta}} \right) \right) \tag{10}
\]

where \( A \), \( B \), \( C \) and \( E \) are constant coefficients that depend only on the physical construction and parameters associated with the pendulum. For the simulation, we are going to use values of these constants provided in the reference [2]; which are \( A = 110 \text{ s}^{-2} \), \( B = 0.01 \text{ s}^{-2} \), \( C = 0.001 \text{ s}^{-1} \), \( E = 0.2 \text{ m}^2\text{s}^{-2} \), \( \theta(0) = 0.2 \text{ rad} \) and \( \omega(0) = 0 \). All these values and conditions will be kept constant throughout all simulation runs. Remember that our pendulum will have different values. We, the authors, will appreciate if some students could actually estimate these parameters for our very pendulum.

Download the file named pendode1.m file from the software codes link on the experiment web page https://www.physlab.org/experiment/the-magnetic-pendulum/. Open the m-file and you will see the following:

```matlab
function dy = pendode1(t,y)
dy = zeros(2,1);
A=;
B=;
C=;
E=;
g=;\text{V} L = (3*g)/(2*A);\text{V} d=80e-3;
```
dy(1)=;
dy(2)=;
end

Now here is one possible recipe for using this file to simulate the behavior of your nonlinear
pendulum.

1. The file pendode1.m file should be saved in the current working directory of Matlab.

2. Insert your values for the parameters A, B, C, E and g into pendode1.m.

3. In our file pendode1.m, we have defined the two elements in the vectors labeled y and
dy as y(1) ≡ ω, y(2) ≡ θ, dy(1) ≡ ∆ω and dy(2) ≡ ∆θ. Write the two Equations (7) and
(8), respectively, in front of dy(1)=; and dy(2)=; For example for writing ∆θ = 10ω−2
you will write dy(2)=10y(1)−2;.

4. Set the value of d for which want to run your simulation. After the parameter values
have been set and the equations have been defined you will now initiate the simulation
by typing in the following commands in Matlab command window:

    options = odeset('RelTol',2.22045e-14,'AbsTol',[1e-14 1e-14]);
    [T ,Y]=(@pendode1,[0 1000],[0 0.2],options);

Here is a brief description of these two lines of code: ode113 is a solver, called on
to solve the system of differential equations which we have defined in pendode1.m.
The interval [0 1000] indicates the time in seconds (or the values of time vector, T)
over which the solver will solve the differential equations defined in Eqs. (7) and (8).
Similarly, the vector [0 0.2] indicates the initial conditions of the variables y(1) and
y(2).

The options command sets the relative and absolute tolerance levels of the ode solver
for our two parameters: ω, θ. As we are looking at a very sensitive pendulum system
therefore the tolerance levels have been set extremely low to give us a high degree of
accuracy in our results. These tolerance levels have been adjusted after trials.

After finishing off its processing the ode solver will return to you two vectors in the
Matlab workspace: a time vector T and another vector Y which comprises two columns,
each corresponding to our two variables, ω and θ.

5. Plot the time series of θ.

6. Plot the time series of ω.
7. Make a phase portrait showing the variation of $\omega$ versus $\theta$.

A set of sample results for varying distances is shown in Figure 3.

3.2.3 Experimental determination of the resonance frequency

This section of the experiment will use Physlab’s computer program magpendulum.exe. We will observe the behavior of the pendulum under the action of varying the distance $d$ between the magnet and the magnetic tip of the pendulum. The motor will be driven at about 70–80% of the pendulum’s resonance frequency $\Omega_0$. So the first step is to actually determine the resonance frequency. This can in fact be quite easily measured. Vary the motor frequency $\Omega$ in small steps from the user interface of magpendulum.exe and note where the amplitude is maximized. The software, very conveniently, shows the maximum in $\theta$. One could also plot the variation of the maximum amplitude with $\Omega$. For the remainder of the experiment, the motor frequency will be fixed slightly below the resonance frequency.

3.2.4 Experimental observation of the nonlinear motion and development of the phase portraits

After setting $\Omega \approx 0.7\Omega_0$, we will now vary $d$ in the same way as we did for the simulation. In the practical scenario, however, as the smallest unit on the measuring tape is only around 0.1 cm, therefore we will not achieve the precision we enjoyed in the simulation. Nevertheless, our pendulum should behave in exactly the same way. The distance $d$ is varied by adjusting the height of the post on which the magnet is perched. We remember that the initial conditions don’t change for the various experimental runs. For example, our driver should start from right above the magnet and that the pendulum should be static when you start driving it. Some sample results are show in Figure 4. Their exploded views can be seen on our website https://www.physlab.org/experiment/the-magnetic-pendulum/.
Figure 3: The results of our simulations showing the phase portraits ((a),(c),(e),(g)) and the time series of the angles ((b),(d),(f),(h)). The distance between the magnets $d$ is decreasing as we are going down this panel. In (c) and (d), the pendulum exhibits chaotic motion and then gets stuck towards one side of the magnet. It is attracted into an “attractor”! In (e) through (h), the pendulum is unable to overcome the attractive force and remains glued inside the right attractor. Note the close correspondence between the phase portraits and the time series.
Figure 4: Some representative experimental results showing the phase portraits ((a),(c),(e),(g)) and the time series of the angles ((b),(d),(f),(h)). The distance between the magnets \( d \) is decreasing as we are going down from (a) to (f). The pair (e),(f) and (g),(h) show results while keeping the distance \( d \) fixed but changing the initial conditions \( \theta_0 \).