Assignment 6: No Longer Frozen
Due Date: December 9, 2019

1. **Art, Lenny, and Hilbert (15 Points)**

   The following is an excerpt from *Quantum Mechanics: The Theoretical Minimum* by Leonard Susskind.

   Art and Lenny expected some action at Hilbert's Place. But all the state-vectors were absolutely still—frozen, you might say.

   Lenny: *This is boring, Art. Doesn't anything ever happen around here? Hey Hilbert, why is this joint so still?*

   Hilbert: *Oh, don't worry. Things will pick up as soon as the Hamiltonian gets here.*

   Art: *The Hamiltonian? He sounds like a real operator.*

   Now that you have learnt what the Hamiltonian is, you are capable of working out the time evolution of quantum states. Therefore, as of now, you would not be dealing solely with states that are frozen in time.

   (a) **(10 Points)** A spin-1 particle with magnetic moment $\mu = \frac{gq}{2m}\mathbf{S}$ is situated in magnetic field $\mathbf{B} = B_0\,\hat{k}$, where g is the Lande g-factor, q is charge of the particle, and m is its mass. At time t_0, the particle is in the state with $S_y = \hbar$. Determine the state of the particle at time t. Calculate how $\langle \hat{S}_y \rangle$ varies with time.

   (b) **(5 Points)** Verify the following:

   $$\frac{d\langle \hat{S}_y \rangle}{dt} = \frac{i}{\hbar}\,\langle [\hat{H}, \hat{S}_y] \rangle,$$

   where $\hat{H}$ is the Hamiltonian for the system in part a.

2. **New Operator, Old Games (20 Points)**

   An X-ray fluorescence spectrometer analyzes the energy of an X-ray photon. Energy is an observable, and the corresponding Hermitian operator is $\hat{H}$. The incoming state is

   $$|\psi\rangle = \sqrt{2}\,|1\rangle + \sqrt{3}\,|2\rangle + |3\rangle + |4\rangle,$$

   where $|1\rangle$, $|2\rangle$, $|3\rangle$, and $|4\rangle$ are the nondegenerate eigenstates of $\hat{H}$ such that

   $$\hat{H}|n\rangle = n^2 \varepsilon_0 |n\rangle$$

and ε_0 is a constant with dimensions of energy.

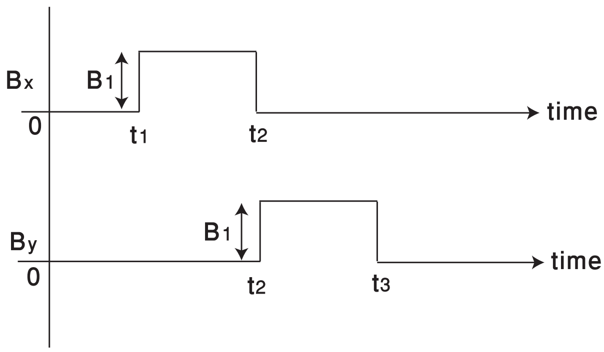
(a) **(5 Points)** Normalize the input state, $|\psi\rangle$.

(b) **(5 Points)** Write $\hat{H}$ as a sum of outer products.

(c) **(5 Points)** What are the possible outcomes of the measurement of energy, and what are their respective probabilities?

(d) **(5 Points)** What is the average energy measured when the analysis is repeated on a large number of identically prepared X-ray photons?

3. **Holding the Reins (10 Points)**



A systematic application of appropriate magnetic fields grants one exquisite control of the direction that the spin of an electron points in. In this problem, you would get to appreciate what I have just said.

A spin-$\frac{1}{2}$ particle of gyromagnetic ratio γ is initialized in the state $|+z\rangle$, which is an eigenstate of the $\hat{S}_z$ operator. Static magnetic fields are applied along the x and y directions for the durations shown in the figure. Referring to the figure, find the quantum state at time t_3.

4. **Time Evolution: A Warm-Up (10 Points)**

A magnetic field pointing in the z direction corresponds to the Hamiltonian $\omega_o\hat{S}_z$. A spin-1 particle is placed in the field. Its initial state, represented in the eigenbasis of $\hat{S}_z$, is

$$\begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}.$$

(a) **(5 Points)** What is the state at time t? Do not forget to mention your basis if you write a matrix representation.

(b) **(5 Points)** After time t, $\hat{S}_x$ is measured. What is the probability that zero is obtained as the measurement outcome?

5. **Final Fun (10 Points)**

Written in the basis $\{|1\rangle, |2\rangle, |3\rangle\}$, the Hamiltonian of a three-level system is

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_0 & 0 \\ A & 0 & E_0 \end{pmatrix}.$$

If the system is in the state $|3\rangle$ at time 0 s, how long would it take for the system to be in the state

$$|\psi\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}?$$