Quantum Mechanics 1: Drill Questions

Solution

Q 1

(a) We have the state,

$$|\psi\rangle = cos(\theta/2)|z\rangle + e^{i\phi} sin(\theta/2)|-z\rangle.$$

The probability will be ,

$$\begin{aligned}
P\left(\hat{S}_z = -\frac{\hbar}{2}||\psi\rangle\right) &= \left|\langle -z|\psi\rangle\right|^2, \\
&= \left|\langle -z|\left(cos(\theta/2)|z\rangle + e^{i\phi} sin(\theta/2)|-z\rangle\right)\right|^2, \\
&= \left|\langle -z|cos(\theta/2)|z\rangle + \langle -z|e^{i\phi} sin(\theta/2)|-z\rangle\right|^2, \\
&= \left|e^{i\phi} sin(\theta/2)\right|^2, \\
&= sin^2(\theta/2).
\end{aligned}$$

(b)

$$\begin{aligned}
\langle \hat{S}_z\rangle &= \langle\psi|\hat{S}_z|\psi\rangle, \\
&= \left(cos(\theta/2)\langle z| + e^{-i\phi} sin(\theta/2)\langle -z|\right)\hat{S}_z\left(cos(\theta/2)|z\rangle + e^{i\phi} sin(\theta/2)|-z\rangle\right), \\
&= \frac{\hbar}{2}\left(\cos^2\theta/2 - \sin^2\theta/2\right), \\
&= \frac{\hbar}{2}\cos\theta
\end{aligned}$$

and

$$\begin{aligned}
\langle \hat{S}_z{}^2\rangle &= \langle\psi|\hat{S}_z{}^2|\psi\rangle, \\
&= \left(cos(\theta/2)\langle z| + e^{-i\phi} sin(\theta/2)\langle -z|\right)\hat{S}_z{}^2\left(cos(\theta/2)|z\rangle + e^{i\phi} sin(\theta/2)|-z\rangle\right), \\
&= \frac{\hbar^2}{4}\left(\cos^2\theta/2 + \sin^2\theta/2\right),
\end{aligned}$$

$$= \frac{\hbar^2}{4}$$

So the variance will be,

$$\begin{aligned}
\Delta S_z &= \sqrt{\langle \hat{S_z}^2 \rangle - \langle \hat{S}_z \rangle^2}, \\
&= \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{4}\cos^2\theta}, \\
&= \frac{\hbar}{2}\sin\theta
\end{aligned}$$

Q 2

(a) A similarity matrix $\hat{S}$ that takes a state vector from the $\{|v_e\rangle, |v_\mu\rangle\}$ (weak basis) to the $\{|v_1\rangle, |v_2\rangle\}$ (mass basis) is given by

$$\hat{S} = \begin{pmatrix} \langle v_1|v_e\rangle & \langle v_1|v_\mu\rangle \\ \langle v_2|v_e\rangle & \langle v_2|v_\mu\rangle \end{pmatrix} = \begin{pmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{pmatrix}.$$

Using Eqs (1) and (2), the matrix elements are

$$\begin{aligned}
\hat{S}_{11} &= \cos\frac{\theta}{2} \qquad \hat{S}_{12} = \sin\frac{\theta}{2}, \\
\hat{S}_{21} &= \sin\frac{\theta}{2} \qquad \hat{S}_{22} = -\cos\frac{\theta}{2}.
\end{aligned}$$

Hence the similarity matrix $\hat{S}$ is

$$\hat{S} = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ \sin\theta/2 & -\cos\theta/2 \end{pmatrix}.$$

(b) The initial quantum state of an electron neutrino created on the sun is

$$|\psi(0)\rangle = |v_e\rangle = \cos\frac{\theta}{2}|v_1\rangle + \sin\frac{\theta}{2}|v_2\rangle.$$

At time t, the state becomes

$$|\psi(t)\rangle = e^{-i\hat{\mathcal{H}}t/\hbar}|\psi(0)\rangle = e^{-i\hat{\mathcal{H}}t/\hbar}\left(\cos\frac{\theta}{2}|v_1\rangle + \sin\frac{\theta}{2}|v_2\rangle\right).$$

Here $\hat{\mathcal{H}}$ is the mass Hamiltonian with eigenstates $|v_1\rangle$ and $|v_2\rangle$ and eigenvalues E_1 and E_2

$$|\psi(t)\rangle = \cos\frac{\theta}{2}\, e^{-iE_1t/\hbar}|v_1\rangle + \sin\frac{\theta}{2}\, e^{-iE_2t/\hbar}|v_2\rangle$$

(c) The probability of detecting a muon neutrino at time $t = \tau$ is

$$\begin{aligned}
P(\mu,\tau) &= \left|\langle v_\mu|\psi(\tau)\rangle\right|^2 \\
\langle v_\mu|\psi(\tau)\rangle &= \left(\sin\frac{\theta}{2}\langle v_1| - \cos\frac{\theta}{2}\langle v_2|\right)\left(\cos\frac{\theta}{2}\, e^{-iE_1\tau/\hbar}|v_1\rangle + \sin\frac{\theta}{2}\, e^{-iE_2\tau/\hbar}|v_2\rangle\right) \\
&= \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-iE_1\tau/\hbar} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-iE_2\tau/\hbar} \\
&= \frac{1}{2}\sin\theta\left(e^{-iE_1\tau/\hbar} - e^{-iE_2\tau/\hbar}\right),
\end{aligned}$$

where we have used the half angle formula $\sin\theta = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$.

$$\begin{aligned}
\Rightarrow \left|\langle v_\mu|\psi(\tau)\rangle\right|^2 &= \frac{1}{4}\sin^2\theta\left(e^{iE_1\tau/\hbar} - e^{iE_2\tau/\hbar}\right)\left(e^{-iE_1\tau/\hbar} - e^{-iE_2\tau/\hbar}\right) \\
&= \frac{1}{4}\sin^2\theta\left(1 + 1 - e^{i(E_1-E_2)\tau/\hbar} - e^{-i(E_1-E_2)\tau/\hbar}\right) \\
&= \frac{1}{4}\sin^2\theta\left(2 - 2\cos\left(\frac{(E_1-E_2)\tau}{\hbar}\right)\right) \\
&= \frac{1}{2}\sin^2\theta\left(1 - \cos\left(\frac{(E_1-E_2)\tau}{\hbar}\right)\right) \\
&= \sin^2\theta\sin^2\left(\frac{(E_1-E_2)\tau}{2\hbar}\right),
\end{aligned}$$

where $\sin^2(\theta/2) = \frac{1-\cos\theta}{2}$.

(d) The energy corresponds to mass eigenstate $|v_1\rangle$ is given by

$$
\begin{aligned}
E_1 &= \sqrt{((pc)^2 + m_1c^2)^2} \\
&= (pc)\sqrt{1 + \frac{(m_1c^2)^2}{(pc)^2}}.
\end{aligned}
$$

For relativistic particles, $(mc^2)/(pc) \ll 1$. So by using Binomial expansion,

$$
\begin{aligned}
E_1 &\approx (pc)\left(1 + \frac{(m_1c^2)^2}{2(pc)^2}\right) \\
&= pc + \frac{m_1^2c^3}{2p}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E_2 &= pc + \frac{m_2^2c^3}{2p} \\
E_1 - E_2 &= \frac{m_1^2c^3}{2p} - \frac{m_2^2c^3}{2p} = \frac{c^3}{2p}(m_1^2 - m_2^2).
\end{aligned}
$$

(e) From part (c) and (d),

$$
\begin{aligned}
P_{v_e \to v_\mu} &= \sin^2\theta \sin^2\left(\frac{c^3}{2p}(m_1^2 - m_2^2)\frac{\tau}{2\hbar}\right) \\
&= \sin^2\theta \sin^2\left(\frac{c^2(m_1^2 - m_2^2)c\tau}{4p\hbar}\right) \\
&= \sin^2\theta \sin^2\left(\frac{c^2(m_1^2 - m_2^2)L}{4p\hbar}\right) \qquad \text{using} \quad L \simeq c\tau
\end{aligned}
$$

(f) When the argument of the $\sin^2$ function changes by $\pi/2$, the probability goes from zero to one. Hence one can find the distance as

$$
\begin{aligned}
\frac{c^2(m_1^2 - m_2^2)\Delta L}{4p\hbar} &= \frac{\pi}{2} \\
\Delta L &= \frac{2\pi p\hbar}{c^2(m_1^2 - m_2^2)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{ph}{c^2(m_1^2 - m_2^2)} \\
&= \frac{Eh}{c^3(m_1^2 - m_2^2)} \qquad \text{using } \ p = E/c \\
&= \frac{(8\times10^6)\cancel{(1.602\times10^{-19})}(6.63\times10^{-34})}{\cancel{(3\times10^8)^3}(8\times10^{-5})(1.602\times10^{-19})^{\cancel{2}}/(3\times10^8)^{\cancel{4}}} \\
&\simeq 124 \text{ km.}
\end{aligned}
$$

Q 3

(a) We can write the given state as,

$$|\psi\rangle = \frac{1}{2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle$$

So the state after some later time t will be,

$$
\begin{aligned}
e^{-i\hat{H}t/\hbar}|\psi\rangle &= e^{-i\hat{H}t/\hbar}\left(\frac{1}{2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle\right), \\
&= \frac{1}{2}e^{-i\hat{H}t/\hbar}|1,1\rangle + \frac{1}{\sqrt{2}}e^{-i\hat{H}t/\hbar}|1,0\rangle + \frac{1}{2}e^{-i\hat{H}t/\hbar}|1,-1\rangle, \\
&= \frac{1}{2}e^{-i\omega_o t/2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}e^{+i\omega_o t/2}|1,-1\rangle,
\end{aligned}
$$

(b) Let's find the eigenbasis of $\hat{S}_x$,

$$\frac{\hbar}{\sqrt{2}}\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0,$$

$$
\begin{aligned}
&= \frac{\hbar}{\sqrt{2}}\left(-\lambda(\lambda^2-1) - 1(-\lambda)\right), \\
&= \frac{\hbar}{\sqrt{2}}(-\lambda)\left(\lambda^2 - 1 - 1\right), \\
&= \frac{\hbar}{\sqrt{2}}(-\lambda)\left(\lambda^2 - 2\right) = 0,
\end{aligned}
$$

So we get from this equation $\lambda = 0, \pm\hbar$. Now the eigenvalue equation will be,

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

$$\begin{aligned} \frac{\hbar}{\sqrt{2}}b &= \lambda a, \\ \frac{\hbar}{\sqrt{2}}(a+c) &= \lambda b, \\ \frac{\hbar}{\sqrt{2}}b &= \lambda c, \end{aligned}$$

<u>For $\lambda = \hbar$</u>

$$\begin{aligned} \frac{\hbar}{\sqrt{2}}b &= \hbar a, \\ \frac{\hbar}{\sqrt{2}}(a+c) &= \hbar b, \\ \frac{\hbar}{\sqrt{2}}b &= \hbar c, \end{aligned}$$

This will leads to $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$, which will be normalized to $\frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$.

<u>For $\lambda = 0$</u>

$$\begin{aligned} \frac{\hbar}{\sqrt{2}}b &= 0, \\ \frac{\hbar}{\sqrt{2}}(a+c) &= 0, \\ \frac{\hbar}{\sqrt{2}}b &= 0, \end{aligned}$$

This will leads to $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, which will be normalized to $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

<u>For $\lambda = -\hbar$</u>

$$\begin{aligned}
\frac{\hbar}{\sqrt{2}} b &= -\hbar a, \\
\frac{\hbar}{\sqrt{2}} (a+c) &= -\hbar b, \\
\frac{\hbar}{\sqrt{2}} b &= -\hbar c,
\end{aligned}$$

This will leads to $\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$, which will be normalized to $\frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$.

Eigenstates	Eigenvalues
$\vert\lambda_1\rangle = \frac{1}{2}\left(\vert 1,1\rangle + \sqrt{2}\vert 1,0\rangle + \vert 1,-1\rangle\right)$	$\hbar$
$\vert\lambda_2\rangle = \frac{1}{\sqrt{2}}\left(-\vert 1,1\rangle + \vert 1,-1\rangle\right)$	0
$\vert\lambda_3\rangle = \frac{1}{2}\left(\vert 1,1\rangle - \sqrt{2}\vert 1,0\rangle + \vert 1,-1\rangle\right)$	$-\hbar$

So the probability of find the zero as the measurement out come will be,

$$\begin{aligned}
\left|\langle\lambda_2|\psi(t)\rangle\right|^2 &= \left|\frac{1}{\sqrt{2}}\left(-\langle 1,1| + \langle 1,-1|\right)\left(\frac{1}{2}e^{-i\omega_o t/2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}e^{+i\omega_o t/2}|1,-1\rangle\right)\right|^2, \\
&= \left|\frac{-1}{2\sqrt{2}}e^{-i\omega_o t/2} + \frac{1}{2\sqrt{2}}e^{i\omega_o t/2}\right|^2, \\
&= \frac{1}{8}\left|2i\sin(\omega_o t/2)\right|^2. \\
&= \frac{1}{2}\sin^2(\omega_o t/2)
\end{aligned}$$