Quiz #3
(Solution)

(1)(a) $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \Rightarrow \hat{L}_x^2 + \hat{L}_y^2 = \hat{L}^2 - \hat{L}_z^2.$

$$(\hat{L}_x^2 + \hat{L}_y^2)|l, m\rangle = (\hat{L}^2 - \hat{L}_z^2)|l, m\rangle$$

$$= \hat{L}^2|l, m\rangle - \hat{L}_z^2|l, m\rangle$$

$$= (\hbar^2(l)(l+1) - \hbar^2 m^2)|l, m\rangle$$

$$= \hbar^2(l^2 + l - m^2)|l, m\rangle$$

$\therefore$ the measured value is $\hbar^2(l^2 + l - m^2)$.

(b) $\langle \hat{L}_x\rangle = \frac{1}{2}\langle\psi|(\hat{L}_+ + \hat{L}_-)|\psi\rangle = \frac{1}{4}(\langle 1, 1| + \langle 1, -1|)(\hat{L}_+ + \hat{L}_-)(|1, 1\rangle + |1, -1\rangle)$

$$= \frac{1}{4}(\langle 1, 1| + \langle 1, -1|)(\hbar\sqrt{2}\,|1, 0\rangle + \hbar\sqrt{2}\,|1, 0\rangle)$$

$$= \hbar(\langle 1, 1| + \langle 1, -1|)(|1, 0\rangle)\left(\frac{\sqrt{2}}{2}\right)$$

$$= 0.$$

$\langle \hat{L}_y\rangle = \frac{1}{2i}\langle\psi|(\hat{L}_+ - \hat{L}_-)|\psi\rangle = \frac{1}{4i}(\langle 1, 1| + \langle 1, -1|)(\hat{L}_+ - \hat{L}_-)(|1, 1\rangle + |1, -1\rangle)$

$$= \frac{1}{4i}(\langle 1, 1| + \langle 1, -1|)(\hbar\sqrt{2}\,|1, 0\rangle - \hbar\sqrt{2}\,|1, 0\rangle)$$

$$= 0.$$

$\langle \hat{L}_z\rangle = \langle\psi|\hat{L}_z|\psi\rangle = \frac{1}{2}(\langle 1, 1| + \langle 1, -1|)\hat{L}_z(|1, 1\rangle + |1, -1\rangle)$

$$= \frac{1}{2}(\langle 1, 1| + \langle 1, -1|)(\hbar|1, 1\rangle - \hbar|1, -1\rangle)$$

$$= 0.$$

(2)(a) $\langle \hat{J}_z \rangle = \frac{1}{2}\left(\langle \tfrac{1}{2}, \tfrac{1}{2}| + \langle \tfrac{1}{2}, -\tfrac{1}{2}|\right) \hat{J}_z \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right)$

$= \frac{1}{2}\left(\langle \tfrac{1}{2}, \tfrac{1}{2}| + \langle \tfrac{1}{2}, -\tfrac{1}{2}|\right)\left(\frac{\hbar}{2}|\tfrac{1}{2}, \tfrac{1}{2}\rangle - \frac{\hbar}{2}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right)$

$= \frac{1}{2}\left(\frac{\hbar}{2} - \frac{\hbar}{2}\right)$

$= 0.$

(b) Consider the uncertainty relations

$$\Delta J_y \Delta J_z \geq \frac{\hbar}{2} \| \langle \hat{J}_x \rangle \|$$

and

$$\Delta J_z \Delta J_x \geq \frac{\hbar}{2} \| \langle \hat{J}_y \rangle \|.$$

$\| \langle \hat{J}_x \rangle \| = \| \langle \psi | \hat{J}_x | \psi \rangle \| = \frac{1}{4} \left\| \left(\langle \tfrac{1}{2}, \tfrac{1}{2}| + \langle \tfrac{1}{2}, -\tfrac{1}{2}|\right)(\hat{J}_+ + \hat{J}_-) \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right) \right\|$

$= \frac{1}{4} \left\| \left(\langle \tfrac{1}{2}, \tfrac{1}{2}| + \langle \tfrac{1}{2}, -\tfrac{1}{2}|\right)\left(\hbar |\tfrac{1}{2}, \tfrac{1}{2}\rangle + \hbar |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right) \right\|$

$= \frac{1}{4} \| 2\hbar \|$

$= \frac{\hbar}{2}.$

$\| \langle \hat{J}_y \rangle \| = \| \langle \psi | \hat{J}_y | \psi \rangle \| = \frac{1}{4} \left\| \left(\langle \tfrac{1}{2}, \tfrac{1}{2}| + \langle \tfrac{1}{2}, -\tfrac{1}{2}|\right)(\hat{J}_+ - \hat{J}_-) \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right) \right\|$

$= \frac{1}{4} \left\| \left(\langle \tfrac{1}{2}, \tfrac{1}{2}| + \langle \tfrac{1}{2}, -\tfrac{1}{2}|\right)\left(\hbar |\tfrac{1}{2}, \tfrac{1}{2}\rangle - \hbar |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right) \right\|$

$= 0.$

Clearly, $\Delta J_y \Delta J_z \geq \frac{\hbar}{2} \| \hat{J}_x \| = \frac{\hbar}{2}\left(\frac{\hbar}{2}\right) = \frac{\hbar^2}{4} \Rightarrow \Delta J_y > 0.$ This thing implies that we cannot know $\hat{J}_y$. Even though $|\psi\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$ is an eigenstate of $\hat{J}_x$ (compare $|\psi\rangle$ with

$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$), something implying that we know $\hat{J}_x$, we cannot know $\hat{J}_y$ simultaneously. Same goes for $\hat{J}_z$, for $\Delta J_y \Delta J_z \geq \frac{\hbar^2}{4}$ implies that $\Delta J_z > 0$.

(c) The only such state is $|0, 0\rangle$

(3) $[\hat{J}^2, \hat{J}_z]$

$$= [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{J}_z]$$

$$= [\hat{J}_x^2, \hat{J}_z] + [\hat{J}_y^2, \hat{J}_z] + [\hat{J}_z^2, \hat{J}_z]$$

$$= \hat{J}_x[\hat{J}_x, \hat{J}_z] + [\hat{J}_x, \hat{J}_z]\hat{J}_x + \hat{J}_y[\hat{J}_y, \hat{J}_z] + [\hat{J}_y, \hat{J}_z]\hat{J}_y + 0$$

$$= \cancel{-i\hbar\hat{J}_x\hat{J}_y} \cancel{-i\hbar\hat{J}_y\hat{J}_x} + \cancel{i\hbar\hat{J}_y\hat{J}_x} + \cancel{i\hbar\hat{J}_x\hat{J}_y}$$

$= 0$. Q.E.D.