Quiz 3
Time: 1 Hour

1. **Electron Dance (10 Points)**

   The orbital angular momentum state of an electron in the hydrogen atom is $|l, m\rangle$. In classical terms, the electron performs a modest dance by rotating about the hydrogen nucleus.

   (a) **(4 Points)** If we perform an experiment to measure $\hat{L}_x^2 + \hat{L}_y^2$, show that the measured value is $\hbar^2(l^2 + l - m^2)$.

   Now, consider the state

   $$|\psi\rangle = \frac{1}{\sqrt{2}}|1, 1\rangle + \frac{1}{\sqrt{2}}|1, -1\rangle.$$

   If you found the electron dance in part a boring, just realize that a hydrogen electron in the state $|\psi\rangle$ is in a superposition of rotating both clockwise and anticlockwise about the nucleus, a routine which even Michael Jackson would have failed to perform.

   (b) **(6 Points)** Assuming that our electron is in the state $|\psi\rangle$, calculate $\langle \hat{L}_x \rangle$, $\langle \hat{L}_y \rangle$, and $\langle \hat{L}_z \rangle$.
   Hint: Use $\hat{L}_+$ and $\hat{L}_-$.

2. **An Uncertainty Puzzle (10 Points)**

   Nature seems to hate our knowing any two components of the angular momentum simultaneously, something which is manifest in the fact that no two of the operators $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ commute. However, consider the state

   $$|\psi\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle.$$

   (a) **(2 Points)** For this state, show that $\langle \hat{J}_z \rangle = 0$.

   Now, notice that

   $$\Delta J_x \Delta J_y \geq (\hbar/2)|\langle \hat{J}_z \rangle| = 0.$$

(b) **(6 Points)** Does this uncertainty relation imply that we can know $\hat{J}_x$ and $\hat{J}_y$ simultaneously? If not, explain your answer.

Hint: The given uncertainty relation is not the only one: there are two more.

(c) **(2 Points)** Is there an angular momentum state that ΔJ_x, ΔJ_y, and ΔJ_z all vanish for? If so, which state is it?

3. **Crumbs (5 Points)**

Use

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

to show that $[\hat{J}^2, \hat{J}_z] = 0$.

Hint: Use $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ as well.