Assignment #5

(Solution)

(1)(a) $\hat{J}^2 \doteq \begin{bmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{bmatrix}.$

$\hat{J}_z \doteq \begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix}.$ Basis $= \{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}.$

$\hat{J}_+ \doteq \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

$\hat{J}_- = \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$

$\hat{J}_x = \dfrac{\hat{J}_+ + \hat{J}_-}{2} \doteq \dfrac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

$\hat{J}_y = \dfrac{\hat{J}_+ - \hat{J}_-}{2i} \doteq \dfrac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}.$

(b) $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$

(c) $[\hat{J}_x, \hat{J}_y] \doteq \dfrac{\hbar^2}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} - \dfrac{\hbar^2}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$= \dfrac{\hbar^2}{2}\begin{bmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{bmatrix} - \dfrac{\hbar^2}{2}\begin{bmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{bmatrix}$

$$= \frac{\hbar^2}{2}\begin{bmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{bmatrix}$$

$$= i\hbar\begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix}.$$

$$[\hat{J}_y, \hat{J}_z] \doteq \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$= \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{bmatrix} - \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{bmatrix}$$

$$= \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

$$= i\hbar\begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix}.$$

$$[\hat{J}_z, \hat{J}_x] \doteq \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}$$

$$-\frac{\hbar^2}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}$$

$$= \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix}0&1&0\\0&0&0\\0&-1&0\end{bmatrix} - \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix}0&0&0\\1&0&-1\\0&0&0\end{bmatrix}$$

$$= \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix}0&1&0\\-1&0&1\\0&-1&0\end{bmatrix}$$

$$= i\hbar\begin{bmatrix}0&-\frac{i\hbar}{\sqrt{2}}&0\\\frac{i\hbar}{\sqrt{2}}&0&-\frac{i\hbar}{\sqrt{2}}\\0&\frac{i\hbar}{\sqrt{2}}&0\end{bmatrix}.$$

(2)(a) Possible measurement results are eigenvalues of $\hat{J}_y$: $\hbar$, 0, and $-\hbar$.

(b) For $J_y = \hbar$, we can calculate the following eigenvector:

$$|\psi\rangle \doteq \frac{1}{2}\begin{bmatrix}1\\\sqrt{2}\,i\\-1\end{bmatrix}.$$

$$\langle\hat{J}_z\rangle \doteq \frac{1}{4}\begin{bmatrix}1&-\sqrt{2}\,i&-1\end{bmatrix}\begin{bmatrix}\hbar&0&0\\0&0&0\\0&0&-\hbar\end{bmatrix}\begin{bmatrix}1\\\sqrt{2}\,i\\-1\end{bmatrix}$$

$$\langle\hat{J}_z\rangle \doteq \frac{\hbar}{4}\begin{bmatrix}1&-\sqrt{2}\,i&-1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = 0.$$

$$\langle \hat{J}_z^2 \rangle \doteq \frac{1}{4}\begin{bmatrix} 1 & -\sqrt{2}i & -1 \end{bmatrix}\begin{bmatrix} \hbar^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar^2 \end{bmatrix}\begin{bmatrix} 1 \\ \sqrt{2}i \\ -1 \end{bmatrix}$$

$$\langle \hat{J}_z^2 \rangle \doteq \frac{1}{4}\begin{bmatrix} 1 & -\sqrt{2}i & -1 \end{bmatrix}\begin{bmatrix} \hbar^2 \\ 0 \\ -\hbar^2 \end{bmatrix} = \frac{\hbar^2}{2}.$$

$$\Delta J_z = \sqrt{\langle \hat{J}_z^2 \rangle - \langle J_z \rangle^2}$$

$$\Delta J_z = \sqrt{\frac{\hbar^2}{2} - 0^2}$$

$$\Delta J_z = \frac{\hbar}{\sqrt{2}}.$$

(3)

$$\hat{A} = \frac{1}{2}(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x).$$

$$\langle \hat{A} \rangle = \langle j, m|\hat{A}|j, m\rangle$$

$$\langle \hat{A} \rangle = \frac{1}{2}\langle j, m|(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x)|j, m\rangle$$

$$\langle \hat{A} \rangle = \frac{1}{2}\langle j, m|\left(\left(\frac{\hat{J}_+ + \hat{J}_-}{2}\right)\left(\frac{\hat{J}_+ - \hat{J}_-}{2i}\right) + \left(\frac{\hat{J}_+ - \hat{J}_-}{2i}\right)\left(\frac{\hat{J}_+ + \hat{J}_-}{2}\right)\right)|j, m\rangle$$

$$\langle \hat{A} \rangle = \frac{1}{2}\langle j, m|\left(\frac{\hat{J}_+\hat{J}_+ - \hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+ - \hat{J}_-\hat{J}_-}{4i} \quad \frac{\hat{J}_+\hat{J}_+ + \hat{J}_+\hat{J}_- - \hat{J}_-\hat{J}_+ - \hat{J}_-\hat{J}_-}{4i}\right)|j, m\rangle$$

$$\langle \hat{A} \rangle = \frac{1}{2}\langle j, m|\left(\frac{\hat{J}_+\hat{J}_+ - \hat{J}_-\hat{J}_-}{4i}\right)|j, m\rangle = 0.$$

$$\langle \hat{A}^2 \rangle = \langle j, m | \hat{A}^2 | j, m \rangle$$

$$\langle \hat{A}^2 \rangle = \frac{1}{4} \langle j, m | (\hat{J}_x \hat{J}_y + \hat{J}_y \hat{J}_x)^2 | j, m \rangle$$

$$\langle \hat{A}^2 \rangle = \frac{1}{4} \langle j, m | \left(\frac{\hat{J}_+ \hat{J}_+ - \hat{J}_- \hat{J}_-}{4i} \right)^2 | j, m \rangle$$

$$\langle \hat{A}^2 \rangle = -\frac{1}{64} \langle j, m | (\hat{J}_+ \hat{J}_+ - \hat{J}_- \hat{J}_-)^2 | j, m \rangle$$

$$\langle \hat{A}^2 \rangle = -\frac{1}{64} \langle j, m | (\hat{J}_+ \hat{J}_+ - \hat{J}_- \hat{J}_-)(\hat{J}_+ \hat{J}_+ - \hat{J}_- \hat{J}_-) | j, m \rangle$$

$$\langle \hat{A}^2 \rangle = -\frac{1}{64} \langle j, m | (\hat{J}_+ \hat{J}_+ \hat{J}_+ \hat{J}_+ - \hat{J}_+ \hat{J}_+ \hat{J}_- \hat{J}_- - \hat{J}_- \hat{J}_- \hat{J}_+ \hat{J}_+ + \hat{J}_- \hat{J}_- \hat{J}_- \hat{J}_-) | j, m \rangle$$

$$\langle \hat{A}^2 \rangle = -\frac{1}{64} \langle j, m | (- \hat{J}_+ \hat{J}_+ \hat{J}_- \hat{J}_- - \hat{J}_- \hat{J}_- \hat{J}_+ \hat{J}_+) | j, m \rangle$$

$$\langle \hat{A}^2 \rangle = -\frac{1}{64} \begin{pmatrix} \left(-\hbar\sqrt{j(j+1) - m(m-1)}\right)\left(\hbar\sqrt{j(j+1) - (m-1)(m-2)}\right) \\ \left(\hbar\sqrt{j(j+1) - (m-2)(m-1)}\right)\left(\hbar\sqrt{j(j+1) - (m-1)(m)}\right) \\ -\left(\hbar\sqrt{j(j+1) - m(m+1)}\right)\left(\hbar\sqrt{j(j+1) - (m+1)(m+2)}\right) \\ \left(\hbar\sqrt{j(j+1) - (m+1)(m)}\right)\left(\hbar\sqrt{j(j+1) - (m+2)(m+1)}\right) \end{pmatrix}$$

$$\langle \hat{A}^2 \rangle = -\frac{1}{64} \begin{pmatrix} -\hbar^4 (j(j+1) - m(m-1))(j(j+1) - (m-1)(m-2)) \\ -\hbar^4 (j(j+1) - m(m+1))(j(j+1) - (m+1)(m+2)) \end{pmatrix}$$

$$\langle \hat{A}^2 \rangle = \frac{\hbar^4}{64} \Big((j(j+1) - m(m-1))(j(j+1) - (m-1)(m-2)) + (j(j+1) - m(m+1))(j(j+1) - (m+1)(m+2)) \Big).$$

(4)(a) $\langle \hat{J}_x^2 \rangle = \langle j, m | \left(\frac{\hat{J}_+ + \hat{J}_-}{2} \right)^2 | j, m \rangle$

$$= \frac{1}{4} \langle j, m | (\hat{J}_+ + \hat{J}_-)^2 | j, m \rangle$$

$$= \frac{1}{4} \langle j, m | (\hat{J}_+ + \hat{J}_-)(\hat{J}_+ + \hat{J}_-) | j, m \rangle$$

$$= \frac{1}{4} \langle j, m | (\hat{J}_+ \hat{J}_+ + \hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+ + \hat{J}_- \hat{J}_-) | j, m \rangle$$

$$= \frac{1}{4}\langle j, m|(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+)|j, m\rangle$$

$$= \frac{1}{4}\Big(\hbar^2\sqrt{j(j+1)-(m-1)(m)}\sqrt{j(j+1)-m(m-1)}$$

$$+ \hbar^2\sqrt{j(j+1)-(m+1)(m)}\sqrt{j(j+1)-m(m+1)}\Big)$$

$$= \frac{1}{4}\hbar^2\big(2j(j+1) - m(m-1) - m(m+1)\big)$$

$$= \frac{\hbar^2}{4}\left(2j(j+1) - 2m^2\right)$$

$$= \frac{\hbar^2}{2}(j^2 + j - m^2).$$

$$\langle\hat{J}_x\rangle = \langle j, m|\left(\frac{\hat{J}_+ + \hat{J}_-}{2}\right)|j, m\rangle$$

$$= 0.$$

$$\Delta J_x = \sqrt{\langle\hat{J}_x^2\rangle - \langle\hat{J}_x\rangle^2}$$

$$\Delta J_x = \sqrt{\frac{\hbar^2(j(j+1) - m^2)}{2}}.$$

Since $|j, m\rangle$ is to $\hat{J}_x$ as it is to $\hat{J}_y$, that is, it is characterized by an axial symmetry,

$$\Delta J_y = \sqrt{\frac{\hbar^2(j(j+1) - m^2)}{2}}.$$

Thus,

$$\Delta J_x \Delta J_y = \frac{\hbar^2(j(j+1) - m^2)}{2}. \quad \text{Q.E.D.}$$

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(b)

$$\Delta J_x \Delta J_y = \sqrt{\frac{\hbar^2(j(j+1)-m^2)}{2}}\sqrt{\frac{\hbar^2(j(j+1)-m^2)}{2}}$$

$$\Delta J_x \Delta J_y = \frac{\hbar^2(j(j+1)-m^2)}{2}.$$

Considering $\hat{J}_+|j,m\rangle$, we have the following:

$$(\hat{J}_+|j,m\rangle)^\dagger(\hat{J}_+|j,m\rangle) \geq 0$$

$$\langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle \geq 0$$

$$\langle j,m|(\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y)|j,m\rangle \geq 0$$

$$\langle j,m|(\hat{J}_x^2 + i\hat{J}_x\hat{J}_y - i\hat{J}_y\hat{J}_x + \hat{J}_y^2)|j,m\rangle \geq 0$$

$$\langle j,m|(\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)|j,m\rangle \geq 0$$

$$\hbar^2 j(j+1) - \hbar^2 m^2 - \hbar^2 m \geq 0$$

$$\frac{\hbar^2 j(j+1) - \hbar^2 m^2}{2} - \frac{\hbar^2 m}{2} \geq 0$$

$$\frac{\hbar^2(j(j+1)-m^2)}{2} \geq \frac{\hbar^2 m}{2}. \quad \text{Q.E.D.}$$

(5)

$$\text{L.H.S.} = [\hat{A}\hat{B}, \hat{C}]$$

$$= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B}$$

$$= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B}$$

$$= \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + (\hat{A}\hat{C} - \hat{C}\hat{A})\hat{B}$$

$$= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} = \text{R.H.S.}$$

$$[\hat{J}^2, \hat{J}_x] = [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{J}_x]$$

$$= [\hat{J}_x^2, \hat{J}_x] + [\hat{J}_y^2, \hat{J}_x] + [\hat{J}_z^2, \hat{J}_x]$$

$$= 0 + \hat{J}_y[\hat{J}_y, \hat{J}_x] + [\hat{J}_y, \hat{J}_x]\hat{J}_y$$

$$+ \hat{J}_z[\hat{J}_z, \hat{J}_x] + [\hat{J}_z, \hat{J}_x]\hat{J}_z$$

$$= \hat{J}_y(-i\hbar\hat{J}_z) + (-i\hbar\hat{J}_z)\hat{J}_y + \hat{J}_z(i\hbar\hat{J}_y) + (i\hbar\hat{J}_y)\hat{J}_z$$

$$= \cancel{-i\hbar\hat{J}_y\hat{J}_z} \cancel{-i\hbar\hat{J}_z\hat{J}_y} + \cancel{i\hbar\hat{J}_z\hat{J}_y} + \cancel{i\hbar\hat{J}_y\hat{J}_z}$$

$$= 0.$$

$$[\hat{J}^2, \hat{J}_y] = [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{J}_y]$$

$$= [\hat{J}_x^2, \hat{J}_y] + [\hat{J}_y^2, \hat{J}_y] + [\hat{J}_z^2, \hat{J}_y]$$

$$= \hat{J}_x[\hat{J}_x, \hat{J}_y] + [\hat{J}_x, \hat{J}_y]\hat{J}_x + 0 + \hat{J}_z[\hat{J}_z, \hat{J}_y] + [\hat{J}_z, \hat{J}_y]\hat{J}_z$$

$$= \hat{J}_x(\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x) + (\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x)\hat{J}_x$$

$$+ \hat{J}_z(\hat{J}_z\hat{J}_y - \hat{J}_y\hat{J}_z) + (\hat{J}_z\hat{J}_y - \hat{J}_y\hat{J}_z)\hat{J}_z$$

$$= \hat{J}_x(i\hbar\hat{J}_z) + (i\hbar\hat{J}_z)\hat{J}_x$$

$$+ \hat{J}_z(-i\hbar\hat{J}_x) + (-i\hbar\hat{J}_x)\hat{J}_z$$

$$= \cancel{i\hbar\hat{J}_x\hat{J}_z} + \cancel{i\hbar\hat{J}_z\hat{J}_x} \cancel{-i\hbar\hat{J}_z\hat{J}_x} \cancel{-i\hbar\hat{J}_x\hat{J}_z}$$

$$= 0.$$

$$[\hat{J}^2, \hat{J}_z] = [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{J}_z]$$

$$= [\hat{J}_x^2, \hat{J}_z] + [\hat{J}_y^2, \hat{J}_z] + [\hat{J}_z^2, \hat{J}_z]$$

$$= \hat{J}_x[\hat{J}_x, \hat{J}_z] + [\hat{J}_x, \hat{J}_z]\hat{J}_x + \hat{J}_y[\hat{J}_y, \hat{J}_z] + [\hat{J}_y, \hat{J}_z]\hat{J}_y + 0$$

$$= \hat{J}_x(-i\hbar\hat{J}_y) + (-i\hbar\hat{J}_y)\hat{J}_x + \hat{J}_y(i\hbar\hat{J}_x) + (i\hbar\hat{J}_x)\hat{J}_y$$

$$= -i\hbar\cancel{\hat{J}_x\hat{J}_y} - i\hbar\cancel{\hat{J}_y\hat{J}_x} + i\hbar\cancel{\hat{J}_y\hat{J}_x} + i\hbar\cancel{\hat{J}_x\hat{J}_y}$$

$$= 0.$$

(6)(a) $$\hat{J}_+|j, m\rangle = \hbar c_+|j, m+1\rangle$$

$$\langle j, m|\hat{J}_-\hat{J}_+|j, m\rangle = \langle j, m+1|(\hbar^2\|c_+\|^2)|j, m+1\rangle$$

$$\hbar^2\|c_+\|^2 = \langle j, m|(\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y)|j, m\rangle$$

$$\hbar^2\|c_+\|^2 = \langle j, m|(\hat{J}_x^2 + i\hat{J}_x\hat{J}_y - i\hat{J}_y\hat{J}_x + \hat{J}_y^2)|j, m\rangle$$

$$\hbar^2\|c_+\|^2 = \langle j, m|(\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)|j, m\rangle$$

$$\|c_+\|^2 = j(j+1) - m^2 - m$$

$$\|c_+\| = \sqrt{j(j+1) - m^2 - m}$$

Since we take c_+ as a real numbers,

$$c_+ = \sqrt{j(j+1) - m^2 - m}.$$

(b) $\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z.$

It was shown in the previous part.

(7) (a) The allowed values are $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, and $-\frac{3}{2}$.

(b) $\langle\psi|\psi\rangle = 1 \Rightarrow \|N\|^2 \begin{bmatrix} -i & 2 & 3 & -4i \end{bmatrix} \begin{bmatrix} i \\ 2 \\ 3 \\ 4i \end{bmatrix} = 1$

$$\Rightarrow \|N\|^2 (1+4+9+16) = 1$$

$$\Rightarrow \|N\| = \frac{1}{\sqrt{30}}.$$

Following the standard convention, we have the following:

$$N = \frac{1}{\sqrt{30}}.$$

(c) $\hat{S}_+\left|\frac{3}{2}, \frac{3}{2}\right\rangle \doteq \hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{3}{2}\left(\frac{3}{2}+1\right)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

$\hat{S}_+\left|\frac{3}{2}, \frac{1}{2}\right\rangle \doteq \hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right)} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sqrt{3}\hbar \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

$\hat{S}_+\left|\frac{3}{2}, -\frac{1}{2}\right\rangle \doteq \hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2\hbar \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

$\hat{S}_+\left|\frac{3}{2}, -\frac{3}{2}\right\rangle \doteq \hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \left(-\frac{3}{2}\right)\left(-\frac{3}{2}+1\right)} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \sqrt{3}\hbar \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$

$$\hat{S}_+ \doteq \begin{bmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & \sqrt{3}\hbar \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(d) $$\hat{S}_- = \hat{S}_+^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3}\hbar & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{bmatrix}.$$

(e) $$\hat{S}_x \doteq \frac{1}{2}\left(\begin{bmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & \sqrt{3}\hbar \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3}\hbar & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{bmatrix} \right)$$

$$\hat{S}_x \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}.$$

$$\hat{S}_y \doteq \frac{1}{2i}\left(\begin{bmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & \sqrt{3}\hbar \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3}\hbar & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{bmatrix} \right)$$

$$\hat{S}_y \doteq \frac{i\hbar}{2}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}.$$

(f) $$[\hat{S}_x, \hat{S}_y] \doteq \frac{i\hbar^2}{4}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$-\frac{i\hbar^2}{4}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$= \frac{i\hbar^2}{4}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$-\frac{i\hbar^2}{4}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$= \frac{i\hbar^2}{4}\begin{bmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & 1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & -1 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{bmatrix} - \frac{i\hbar^2}{4}\begin{bmatrix} -3 & 0 & -2\sqrt{3} & 0 \\ 0 & -1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & 1 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{bmatrix}$$

$$= \frac{i\hbar^2}{2}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$= i\hbar\begin{bmatrix} \frac{3\hbar}{2} & 0 & 0 & 0 \\ 0 & \frac{\hbar}{2} & 0 & 0 \\ 0 & 0 & -\frac{\hbar}{2} & 0 \\ 0 & 0 & 0 & -\frac{3\hbar}{2} \end{bmatrix}.$$

$$\Rightarrow [\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z. \quad \text{Q.E.D.}$$