Solution Assignment 6 Quantum Mechanics I

1. A spin $-1/2$ particle is in the quantum state

$$|\psi\rangle = cos(\theta/2)|z\rangle + e^{i\phi} sin(\theta/2)|-z\rangle$$

Where $|z\rangle$ and $|-z\rangle$ are eigenstates of the $\hat{S}_z$ operator and θ and ϕ are as defined in the figure.

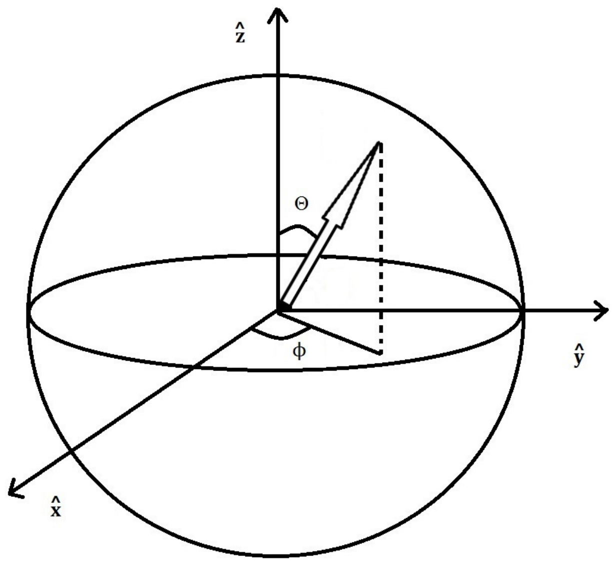


Fig. 1

(a) Suppose that a measurement of $\hat{S}_z$ is carried out. What is the probability that the measurement out come is $-\frac{\hbar}{2}$. [**3**]

(b) Determine the expectation value $\langle \hat{S}_z \rangle$ and the uncertainty $\Delta \hat{S}_z$. [**6**]

Answer

(a) We have the state,

$$|\psi\rangle = cos(\theta/2)|z\rangle + e^{i\phi} sin(\theta/2)|-z\rangle.$$

The probability will be ,

$$\begin{aligned}
P\left(\hat{S}_z = -\frac{\hbar}{2} || \psi\rangle\right) &= \left|\langle -z|\psi\rangle\right|^2, \\
&= \left|\langle -z|\left(cos(\theta/2)|z\rangle + e^{i\phi} sin(\theta/2)|-z\rangle\right)\right|^2, \\
&= \left|\langle -z|cos(\theta/2)|z\rangle + \langle -z|e^{i\phi} sin(\theta/2)|-z\rangle\right|^2, \\
&= \left|e^{i\phi} sin(\theta/2)\right|^2, \\
&= sin^2(\theta/2).
\end{aligned}$$

(b)

$$\begin{aligned}
\langle \hat{S}_z\rangle &= \langle \psi|\hat{S}_z|\psi\rangle, \\
&= \left(cos(\theta/2)\langle z| + e^{-i\phi} sin(\theta/2)\langle -z|\right) \hat{S}_z \left(cos(\theta/2)|z\rangle + e^{i\phi} sin(\theta/2)|-z\rangle\right), \\
&= \frac{\hbar}{2}\left(\cos^2\theta/2 - \sin^2\theta/2\right), \\
&= \frac{\hbar}{2}\cos\theta
\end{aligned}$$

and

$$\begin{aligned}
\langle \hat{S}_z{}^2\rangle &= \langle \psi|\hat{S}_z{}^2|\psi\rangle, \\
&= \left(cos(\theta/2)\langle z| + e^{-i\phi} sin(\theta/2)\langle -z|\right) \hat{S}_z{}^2 \left(cos(\theta/2)|z\rangle + e^{i\phi} sin(\theta/2)|-z\rangle\right), \\
&= \frac{\hbar^2}{4}\left(\cos^2\theta/2 + \sin^2\theta/2\right), \\
&= \frac{\hbar^2}{4}
\end{aligned}$$

So the variance will be,

$$\begin{aligned}
\Delta S_z &= \sqrt{\langle \hat{S}_z{}^2\rangle - \langle \hat{S}_z\rangle^2}, \\
&= \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{4}\cos^2\theta}, \\
&= \frac{\hbar}{2}\sin\theta
\end{aligned}$$

2. The solar neutrino problem was a long outstanding problem in physics. Today, we know that neutrinos are highly relativistic particles with nonzero mass. Neutrinos can occur in two mutually orthogonal states labeled as $|v_e\rangle$ and $|v_\mu\rangle$ corresponding to the two 'flavors' of neutrinos-called the electron neutrino and muon neutrino respectively.

These states are eigenstates of the weak interaction Hamiltonian. However, when neutrinos propagate in free space, the only Hamiltonian of relevance is due to relativistic energy of the particles. The eigenstates of this Hamiltonian are generally called the mass eiegnstates, denoted by $|v_1\rangle$ and $|v_2\rangle$.

Hence states in the two dimensional Hilbert space can be described by the weak interaction eigenstates $\{|v_e\rangle, |v_\mu\rangle\}$ or by the mass eigenstates $\{|v_1\rangle, |v_2\rangle\}$. The relationship between the states is described by

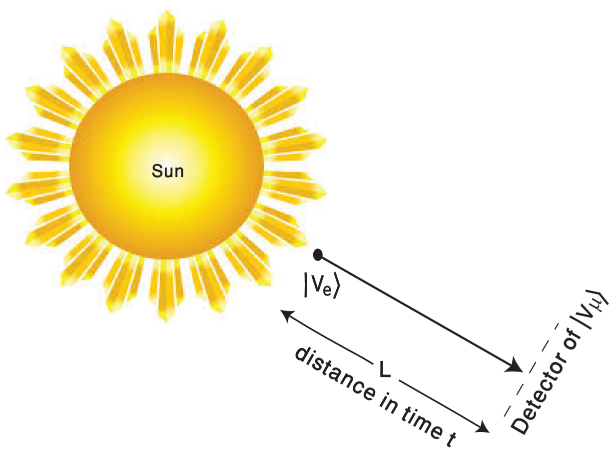
$$|v_e\rangle = \cos\frac{\theta}{2}|v_1\rangle + \sin\frac{\theta}{2}|v_2\rangle \tag{1}$$

$$|v_\mu\rangle = \sin\frac{\theta}{2}|v_1\rangle - \cos\frac{\theta}{2}|v_2\rangle. \tag{2}$$

The angle $\theta/2$ is called the mixing angle.

(a) Write the similarity matrix that takes a state vector from the $\{|v_e\rangle, |v_\mu\rangle\}$ (weak basis) to the $\{|v_1\rangle, |v_2\rangle\}$ (mass basis). [**3**]

(b)



Suppose an electron neutrino $|\psi(0)\rangle = |v_e\rangle$ is created on the sun. In its free propagation towards the earth, only the mass Hamiltonian is operative. The eigenvalues of this Hamiltonian are E_1 and E_2 for the eigenstates $|v_1\rangle$ and $|v_2\rangle$. Write the time-dependent quantum state $|\psi(t)\rangle$ as neutrinos propagate in space [**3**]

(c) Suppose it takes time τ for the neutrino to reach a detector on the earth. The detector can only see muon neutrinos $|v_\mu\rangle$. Find the probability P that the de-

tector detects a muon neutrino, when it is directly facing the stream of neutrinos. (Assume no neutrinos are lost in between.) [**5**]

(d) Neutrinos are relativistic particles. This means that the mass eigenstates have energies

$$\begin{aligned} E_1 &= \sqrt{(m_1c^2)^2 + (pc)^2} \\ E_2 &= \sqrt{(m_2c^2)^2 + (pc)^2}, \end{aligned}$$

where $m_{1,2}$ are masses of the mass eigenstates, c is the speed of light and p is their common momentum. For relativistic particles, $(pc) \gg (mc^2)$. Using Binomial expansion show that [**4**]

$$E_1 - E_2 = \frac{c^3}{2p}(m_1^2 - m_2^2).$$

(e) Using the above expression for $E_1 - E_2$, find the probability $P_{v_e \to v_\mu}$ found in (c), expressed in terms of the distance L. [**2**]

(f) How far will an 8 MeV electron neutrino travel before being converted into a muon neutrino?
Assume, $m_1^2 - m_2^2 \approx 8 \times 10^{-5}$ eV^2/c^4 and use $p = E/c$ to calculate the momentum of the relativistic particle. [**3**]

Answer

(a) A similarity matrix $\hat{S}$ that takes a state vector from the $\{|v_e\rangle, |v_\mu\rangle\}$ (weak basis) to the $\{|v_1\rangle, |v_2\rangle\}$ (mass basis) is given by

$$\hat{S} = \begin{pmatrix} \langle v_1|v_e\rangle & \langle v_1|v_\mu\rangle \\ \langle v_2|v_e\rangle & \langle v_2|v_\mu\rangle \end{pmatrix} = \begin{pmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{pmatrix}.$$

Using Eqs (1) and (2), the matrix elements are

$$\begin{aligned} \hat{S}_{11} &= \cos\frac{\theta}{2} \qquad \hat{S}_{12} = \sin\frac{\theta}{2}, \\ \hat{S}_{21} &= \sin\frac{\theta}{2} \qquad \hat{S}_{22} = -\cos\frac{\theta}{2}. \end{aligned}$$

Hence the similarity matrix $\hat{S}$ is

$$\hat{S} = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ \sin\theta/2 & -\cos\theta/2 \end{pmatrix}.$$

(b) The initial quantum state of an electron neutrino created on the sun is

$$|\psi(0)\rangle = |v_e\rangle = \cos\frac{\theta}{2}|v_1\rangle + \sin\frac{\theta}{2}|v_2\rangle.$$

At time t, the state becomes

$$|\psi(t)\rangle = e^{-i\hat{\mathcal{H}}t/\hbar}|\psi(0)\rangle = e^{-i\hat{\mathcal{H}}t/\hbar}\left(\cos\frac{\theta}{2}|v_1\rangle + \sin\frac{\theta}{2}|v_2\rangle\right).$$

Here $\hat{\mathcal{H}}$ is the mass Hamiltonian with eigenstates $|v_1\rangle$ and $|v_2\rangle$ and eigenvalues E_1 and E_2

$$|\psi(t)\rangle = \cos\frac{\theta}{2}\, e^{-iE_1 t/\hbar}|v_1\rangle + \sin\frac{\theta}{2}\, e^{-iE_2 t/\hbar}|v_2\rangle$$

(c) The probability of detecting a muon neutrino at time $t = \tau$ is

$$\begin{aligned}
P(\mu,\tau) &= \left|\langle v_\mu|\psi(\tau)\rangle\right|^2 \\
\langle v_\mu|\psi(\tau)\rangle &= \left(\sin\frac{\theta}{2}\langle v_1| - \cos\frac{\theta}{2}\langle v_2|\right)\left(\cos\frac{\theta}{2}\, e^{-iE_1\tau/\hbar}|v_1\rangle + \sin\frac{\theta}{2}\, e^{-iE_2\tau/\hbar}|v_2\rangle\right) \\
&= \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-iE_1\tau/\hbar} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-iE_2\tau/\hbar} \\
&= \frac{1}{2}\sin\theta\left(e^{-iE_1\tau/\hbar} - e^{-iE_2\tau/\hbar}\right),
\end{aligned}$$

where we have used the half angle formula $\sin\theta = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$.

$$\begin{aligned}
\Rightarrow \left|\langle v_\mu|\psi(\tau)\rangle\right|^2 &= \frac{1}{4}\sin^2\theta\left(e^{iE_1\tau/\hbar} - e^{iE_2\tau/\hbar}\right)\left(e^{-iE_1\tau/\hbar} - e^{-iE_2\tau/\hbar}\right) \\
&= \frac{1}{4}\sin^2\theta\left(1 + 1 - e^{i(E_1-E_2)\tau/\hbar} - e^{-i(E_1-E_2)\tau/\hbar}\right) \\
&= \frac{1}{4}\sin^2\theta\left(2 - 2\cos\left(\frac{(E_1-E_2)\tau}{\hbar}\right)\right) \\
&= \frac{1}{2}\sin^2\theta\left(1 - \cos\left(\frac{(E_1-E_2)\tau}{\hbar}\right)\right) \\
&= \sin^2\theta\sin^2\left(\frac{(E_1-E_2)\tau}{2\hbar}\right),
\end{aligned}$$

where $\sin^2(\theta/2) = \frac{1-\cos\theta}{2}$.

(d) The energy corresponds to mass eigenstate $|v_1\rangle$ is given by

$$\begin{aligned}
E_1 &= \sqrt{((pc)^2 + m_1c^2)^2} \\
&= (pc)\sqrt{1 + \frac{(m_1c^2)^2}{(pc)^2}}.
\end{aligned}$$

For relativistic particles, $(mc^2)/(pc) \ll 1$. So by using Binomial expansion,

$$
\begin{aligned}
E_1 &\approx (pc)\left(1 + \frac{(m_1c^2)^2}{2(pc)^2}\right) \\
&= pc + \frac{m_1^2 c^3}{2p}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E_2 &= pc + \frac{m_2^2 c^3}{2p} \\
E_1 - E_2 &= \frac{m_1^2 c^3}{2p} - \frac{m_2^2 c^3}{2p} = \frac{c^3}{2p}(m_1^2 - m_2^2).
\end{aligned}
$$

(e) From part (c) and (d),

$$
\begin{aligned}
P_{v_e \to v_\mu} &= \sin^2\theta \sin^2\left(\frac{c^3}{2p}(m_1^2 - m_2^2)\frac{\tau}{2\hbar}\right) \\
&= \sin^2\theta \sin^2\left(\frac{c^2(m_1^2 - m_2^2)c\tau}{4p\hbar}\right) \\
&= \sin^2\theta \sin^2\left(\frac{c^2(m_1^2 - m_2^2)L}{4p\hbar}\right) \qquad \text{using } \; L \simeq c\tau
\end{aligned}
$$

(f) When the argument of the $\sin^2$ function changes by $\pi/2$, the probability goes from zero to one. Hence one can find the distance as

$$
\begin{aligned}
\frac{c^2(m_1^2 - m_2^2)\Delta L}{4p\hbar} &= \frac{\pi}{2} \\
\Delta L &= \frac{2\pi p\hbar}{c^2(m_1^2 - m_2^2)} \\
&= \frac{ph}{c^2(m_1^2 - m_2^2)} \\
&= \frac{Eh}{c^3(m_1^2 - m_2^2)} \qquad \text{using } \; p = E/c \\
&= \frac{(8\times 10^6)\cancel{(1.602\times 10^{-19})}(6.63\times 10^{-34})}{\cancel{(3\times 10^8)^3}(8\times 10^{-5})(1.602\times 10^{-19})^{\cancel{2}}/(3\times 10^8)^{\cancel{4}}} \\
&\simeq 124 \text{ km}.
\end{aligned}
$$

Q3: A magnetic field is pointing in the z-direction corresponding to a hamiltonian $\omega_o \hat{S}_z$. A spin-1 particle is placed inside the field. Its initial state, written in the eigenbasis of $\hat{S}_z$, is:

$$|\psi\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}.$$

Given

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

(a) What is the state after at a time t? Don't forget to mention your basis if you write a matrix representation. [**8 Marks**]

(b) After time t the component $\hat{S}_x$ is measured. What is the probability of obtaining zero as the measurement outcomes? [**7 Marks**]

Answer

(a) We can write the given state as,

$$|\psi\rangle = \frac{1}{2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle$$

So the state after some later time t will be,

$$\begin{aligned} e^{-i\hat{H}t/\hbar}|\psi\rangle &= e^{-i\hat{H}t/\hbar}\left(\frac{1}{2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle\right), \\ &= \frac{1}{2}e^{-i\hat{H}t/\hbar}|1,1\rangle + \frac{1}{\sqrt{2}}e^{-i\hat{H}t/\hbar}|1,0\rangle + \frac{1}{2}e^{-i\hat{H}t/\hbar}|1,-1\rangle, \\ &= \frac{1}{2}e^{-i\omega_o t/2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}e^{+i\omega_o t/2}|1,-1\rangle, \end{aligned}$$

(b) Let's find the eigenbasis of $\hat{S}_x$,

$$
\begin{aligned}
\frac{\hbar}{\sqrt{2}}\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} &= 0, \\
&= \frac{\hbar}{\sqrt{2}}\left(-\lambda(\lambda^2-1)-1(-\lambda)\right), \\
&= \frac{\hbar}{\sqrt{2}}(-\lambda)\left(\lambda^2-1-1\right), \\
&= \frac{\hbar}{\sqrt{2}}(-\lambda)\left(\lambda^2-2\right) = 0,
\end{aligned}
$$

So we get from this equation $\lambda = 0, \pm\hbar$. Now the eigenvalue equation will be,

$$
\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix},
$$

$$
\begin{aligned}
\frac{\hbar}{\sqrt{2}}b &= \lambda a, \\
\frac{\hbar}{\sqrt{2}}(a+c) &= \lambda b, \\
\frac{\hbar}{\sqrt{2}}b &= \lambda c,
\end{aligned}
$$

<u>For $\lambda = \hbar$</u>

$$
\begin{aligned}
\frac{\hbar}{\sqrt{2}}b &= \hbar a, \\
\frac{\hbar}{\sqrt{2}}(a+c) &= \hbar b, \\
\frac{\hbar}{\sqrt{2}}b &= \hbar c,
\end{aligned}
$$

This will leads to $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$, which will be normalized to $\frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$.

<u>For $\lambda = 0$</u>

$$
\begin{aligned}
\frac{\hbar}{\sqrt{2}}b &= 0, \\
\frac{\hbar}{\sqrt{2}}(a+c) &= 0, \\
\frac{\hbar}{\sqrt{2}}b &= 0,
\end{aligned}
$$

This will leads to $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, which will be normalized to $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

<u>For $\lambda = -\hbar$</u>

$$\begin{aligned} \frac{\hbar}{\sqrt{2}} b &= -\hbar a, \\ \frac{\hbar}{\sqrt{2}} (a+c) &= -\hbar b, \\ \frac{\hbar}{\sqrt{2}} b &= -\hbar c, \end{aligned}$$

This will leads to $\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$, which will be normalized to $\frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$.

Eigenstates	Eigenvalues
$\vert\lambda_1\rangle = \frac{1}{2}\left(\vert 1,1\rangle + \sqrt{2}\vert 1,0\rangle + \vert 1,-1\rangle\right)$	$\hbar$
$\vert\lambda_2\rangle = \frac{1}{\sqrt{2}}\left(-\vert 1,1\rangle + \vert 1,-1\rangle\right)$	0
$\vert\lambda_3\rangle = \frac{1}{2}\left(\vert 1,1\rangle - \sqrt{2}\vert 1,0\rangle + \vert 1,-1\rangle\right)$	$-\hbar$

So the probability of find the zero as the measurement out come will be,

$$\begin{aligned} \left|\langle \lambda_2 | \psi(t) \rangle\right|^2 &= \left| \frac{1}{\sqrt{2}} \left(-\langle 1,1| + \langle 1,-1|\right) \left(\frac{1}{2} e^{-i\omega_o t/2} |1,1\rangle + \frac{1}{\sqrt{2}} |1,0\rangle + \frac{1}{2} e^{+i\omega_o t/2} |1,-1\rangle \right) \right|^2, \\ &= \left| \frac{-1}{2\sqrt{2}} e^{-i\omega_o t/2} + \frac{1}{2\sqrt{2}} e^{i\omega_o t/2} \right|^2, \\ &= \frac{1}{8} \left|2i \sin(\omega_o t/2)\right|^2. \\ &= \frac{1}{2} \sin^2(\omega_o t/2) \end{aligned}$$