Tutorial 1 (Solution)

1. **Twirl in the Magnetic Whirl**

A spin-1 particle with magnetic moment $\mu = \frac{gq}{2m}\mathbf{S}$ is situated in magnetic field $\mathbf{B} = B_0\,\hat{k}$, where g is the Lande g-factor, q is charge of the particle, and m is its mass. At time t_0, the particle is in the state with $S_y = \hbar$. Determine the state of the particle at time t. Calculate how $\langle \hat{S}_y \rangle$ varies with time.

Answer

The Hamiltonian describing the interaction of a charged particle with a magnetic field is given by

$$\hat{H} = -\mu \cdot B = -\frac{gq}{2m} B_0 \hat{S}_z = \omega_0 \hat{S}_z,$$

where $\omega_0 = -gq/2m$.

We know that the initial state is an eigenstate of $\hat{S}_y$. We use the same procedure to derive the $\hat{S}_y$ operator and its eigenstate with eigenvalue $\hbar$, as we did in Q 3, HW 7.I am not going to repeat the complete argument here. The final result is that the initial state in the eigenbasis of $\hat{S}_z$ is given by

$$|\psi(0)\rangle \; = \; \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}.$$

Since

$$\begin{aligned} \hat{U}(t) \; &= \; \exp\left(-i\frac{\omega_0 \hat{S}_z t}{\hbar}\right) \\ &= \begin{pmatrix} e^{-i\omega_0 t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{+i\omega_0 t} \end{pmatrix}, \end{aligned}$$

we obtain

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \frac{1}{2}\begin{pmatrix} e^{-i\omega_0 t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{+i\omega_0 t} \end{pmatrix}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\omega_0 t} \\ i\sqrt{2} \\ -e^{+i\omega_0 t} \end{pmatrix},$$

which is the state of the particle at time t. Let's calculate the expectation value of $\hat{S}_y$.

$$\hat{S}_y = \frac{\hbar}{4\sqrt{2}}\begin{pmatrix} e^{+i\omega_0 t} & -i\sqrt{2} & -e^{-i\omega_0 t} \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_0 t} \\ i\sqrt{2} \\ -e^{+i\omega_0 t} \end{pmatrix}$$

$$= \frac{\hbar}{4\sqrt{2}}\begin{pmatrix} e^{+i\omega_0 t} & -i\sqrt{2} & -e^{-i\omega_0 t} \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ ie^{-i\omega_0 t} + ie^{+i\omega_0 t} \\ -\sqrt{2} \end{pmatrix}$$

$$= \hbar\cos\omega_0 t.$$

2. **The Mystery State**

A large number of spin-$\frac{1}{2}$ particles are in the state $|\psi\rangle$.

(a) We measure $\hat{S}_z$ for some of the particles and find that $\frac{3}{4}$ of them are in the state with $S_z = +\frac{\hbar}{2}$. Write the general form of $|\psi\rangle$ satisfying this measurement.

(b) We measure $\hat{S}_y$ for some of the remaining particles and find that $\frac{1}{8}$ of them are in the state with $S_y = -\frac{\hbar}{2}$. Find all the possible states that satisfy both of these measurements.

Answer

(a) The general state for spin- $s = \frac{1}{2}$ particle can be expressed as

$$|\psi\rangle = c_+|{+z}\rangle + c_-|{-z}\rangle,$$

where $|c_+|^2 + |c_-|^2 = 1$ and c_+ and c_- are complex numbers. The condition tells us that $|c_+|^2 = \dfrac{3}{4}$, which implies that $c_+ = \dfrac{\sqrt{3}}{2}e^{i\delta}$ for some δ. We can pick the overall global phase of the state so that $\delta = 0$. This means that $c_- = \frac{1}{2}e^{i\phi}$ and $|\psi\rangle = \frac{\sqrt{3}}{2}|{+z}\rangle + \frac{1}{2}e^{i\phi}|{-z}\rangle$, where $\phi \in [0, 2\pi]$.

(b) As $|\langle -y|\psi\rangle|^2 = \frac{1}{8}$, we know that

$$\langle -y| \;=\; \frac{1}{2}[\langle +z| + i\langle -z|]$$

So

$$\begin{aligned} |\langle -y|\psi\rangle|^2 &= \frac{1}{8} \\ \left|\frac{1}{2}[\langle +z| + i\langle -z|]\left[\frac{\sqrt{3}}{2}|{+z}\rangle + \frac{1}{2}e^{i\phi}|{-z}\rangle\right]\right|^2 &= \frac{1}{8} \\ \frac{1}{8}\left|\sqrt{3} + ie^{i\phi}\right|^2 &= \frac{1}{8} \\ |\sqrt{3} + ie^{i\phi}|^2 &= 1 \\ |\sqrt{3} + i(\cos\phi + i\sin\phi)|^2 &= 1 \\ |\sqrt{3} + i\cos\phi - \sin\phi|^2 &= 1 \\ |(\sqrt{3} - \sin\phi) + i\cos\phi|^2 &= 1 \\ ((\sqrt{3} - \sin\phi)^2 + \cos^2\phi) &= 1 \\ (3 + \sin^2\phi - 2\sqrt{3}\sin\phi + \cos^2\phi) &= 1 \\ (4 - 2\sqrt{3}\sin\phi) &= 1 \end{aligned}$$

$$\begin{aligned} 4 - 1 &= 2\sqrt{3}\sin\phi \\ \frac{\sqrt{3}}{2} &= \sin\phi \end{aligned}$$

hence $\sin\phi = \frac{\sqrt{3}}{2}$. For $0 \le \phi \le 2\pi$ the two possible solutions to this are $\phi = \frac{\pi}{3}$ and $\phi = \frac{2\pi}{3}\cdot$ So we have two possible solutions.

$$|\psi\rangle = \frac{\sqrt{3}}{2}|{+z}\rangle + \frac{1}{2}e^{i\pi/3}|{-z}\rangle,$$

and

$$|\psi\rangle = \frac{\sqrt{3}}{2}|{+z}\rangle + \frac{1}{2}e^{i2\pi/3}|{-z}\rangle\cdot.$$

In order to choose between $\phi = \pi/3$ and $2\pi/3$, $\hat{S}_x$ needs to be measured! So a minimum number of measurements are required to completely determine the state.