16. A Reciprocating Problem

The wheels on the bus go round and round...
The people on the bus go up and down...
—*lyrics from a nursery-school song that has driven the parents of prekindergarten kids nuts for decades*

For an example of the use of trigonometry, geometry (and some calculus, too) to attack an important engineering-physics problem, consider Figure 16.1. There you see a cross-sectional view of a rotating *crankshaft* at A, with a crank arm of length r extending out to a hinged joint at B. As the crankshaft rotates counterclockwise at the constant angular speed of ω radians/second, B rotates along the circumference of a circle with radius r at a constant speed. B, in turn, is linked to a hinged joint at C via a *connecting rod* of length l; C is the location of a *wrist pin* that allows an attached piston to be driven back and forth along the x-axis by the connecting rod.

As described, the piston moves *because* the crankshaft is turned by some external energy source (say, a turbine submerged in running water) and so the entire arrangement could be a pump. On the other hand, the crankshaft could be rotating (and so driving the transmission and, hence, the wheels of a car) *because* the piston is powered by rapidly burning gasoline vapor in a cylinder that encloses the piston. In this case we have an internal combustion engine. In any case, given that the crankshaft is rotating we are to calculate the position, speed, and acceleration of the piston's wrist pin.

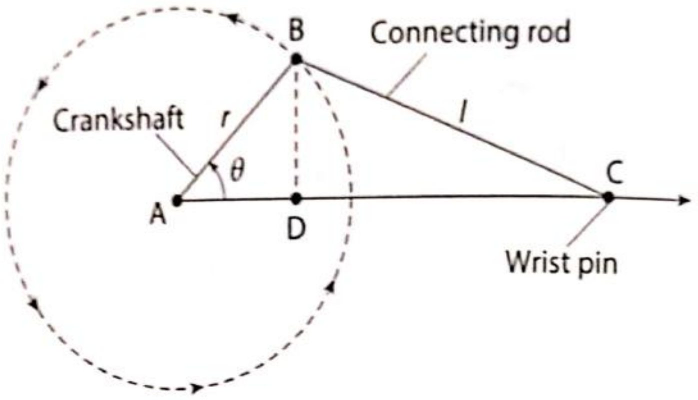

Figure 16.1. Crankshaft/connecting rod/wrist pin geometry

From the geometry shown in Figure 16.1 we can write the wrist pin location, measured from A, as

$$x(t) = \overline{AD} + \overline{DC}.$$

Notice, carefully, that we write $x = x(t)$ because $\theta = \theta(t) = \omega t$. Now, since

$$\overline{AD} = r\cos(\theta),$$

and since the Pythagorean theorem tells us that

$$\overline{BD}^2 + \overline{DC}^2 = l^2,$$

where

$$\overline{BD} = r\sin(\theta),$$

we have

$$x(t) = r\cos(\theta) + \sqrt{l^2 - r^2\sin^2(\theta)} = r\cos(\theta) + l\sqrt{1 - \left(\frac{r}{l}\right)^2 \sin^2(\theta)},$$

or

$$\boxed{\frac{x(t)}{l} = \left(\frac{r}{l}\right)\cos(\theta) + \sqrt{1 - \left(\frac{r}{l}\right)^2 \sin^2(\theta)}, \quad \theta = \omega t.}$$

The boxed equation for $\frac{x(t)}{l}$ shows the useful technique of *normalizing* variables: we have the position of the wrist pin relative to the length of the connecting rod; that is, the length of the connecting rod is playing the role of the unit length.

To find the speed of the wrist pin, we differentiate the expression for $x(t)$—*not* the normalized $\frac{x(t)}{l}$—to get

$$\frac{dx}{dt} = -r\sin(\theta)\frac{d\theta}{dt} + \frac{1}{2}\left\{l^2 - r^2\sin^2(\theta)\right\}^{-1/2}\left\{-2r^2\sin(\theta)\cos(\theta)\frac{d\theta}{dt}\right\},$$

which, after a bit of simple algebra, becomes

$$\frac{dx}{dt} = -\omega r\sin(\theta) - \frac{\omega r r\sin(\theta)\cos(\theta)}{l\sqrt{1-\left(\frac{r}{l}\right)^2\sin^2(\theta)}}.$$

The speed of B is such that in one complete revolution B moves through distance $2\pi r$ in $\frac{2\pi}{\omega}$ seconds, and so B's speed is

$$\frac{2\pi r}{\frac{2\pi}{\omega}} = \omega r,$$

which we'll use as the unit of speed to normalize the speed of the wrist pin. That is, the normalized wrist pin speed is

$$\frac{\frac{dx}{dt}}{\omega r} = -\sin(\theta)\left\{1 + \frac{\left(\frac{r}{l}\right)\sin(\theta)\cos(\theta)}{\sqrt{1-\left(\frac{r}{l}\right)^2\sin^2(\theta)}}\right\}, \quad \theta = \omega t.$$

Finally, to get the acceleration of the wrist pin, we'll differentiate $\frac{dx}{dt}$ to get

$$\frac{d^2x}{dt^2} = -\omega r\cos(\theta)\frac{d\theta}{dt} - r^2\omega \times \left[\frac{\sqrt{l^2-r^2\sin^2(\theta)}\left\{\cos^2(\theta)\frac{d\theta}{dt} - \sin^2(\theta)\frac{d\theta}{dt}\right\} - \sin(\theta)\cos(\theta)\frac{1}{2} \times \left\{l^2-r^2\sin^2(\theta)\right\}^{-1/2}\left\{-2r^2\sin(\theta)\cos(\theta)\frac{d\theta}{dt}\right\}}{l^2-r^2\sin^2(\theta)}\right],$$

which, after a bit of simple algebra, reduces to

$$\frac{d^2x}{dt^2} = -\omega^2 r \left[\cos(\theta) + \left(\frac{r}{l}\right) \frac{\cos(2\theta) + \left(\frac{r}{l}\right)^2 \sin^4(\theta)}{\left\{1 - \left(\frac{r}{l}\right)^2 \sin^2(\theta)\right\}^{3/2}} \right].$$

As we've done twice before, we normalize this acceleration with an acceleration inherent in the problem, and here that is $\omega^2 r$ (which you can check has the units of acceleration;[1] earlier in the book, in Chapter 5, we called this the *centripetal* acceleration). So, the normalized acceleration of the wrist pin is

$$\frac{\frac{d^2x}{dt^2}}{\omega^2 r} = -\left[\cos(\theta) + \left(\frac{r}{l}\right) \frac{\cos(2\theta) + \left(\frac{r}{l}\right)^2 \sin^4(\theta)}{\left\{1 - \left(\frac{r}{l}\right)^2 \sin^2(\theta)\right\}^{3/2}} \right], \quad \theta = \omega t.$$

Figure 16.2 shows plots of our three boxed expressions for the wrist pin's normalized position, speed, and acceleration, for two values of the normalized parameter $\frac{r}{l}$ ($\frac{1}{2}$ in the left column, and $\frac{1}{3}$ in the right column). The independent variable, the angle θ, is plotted on the horizontal axes for one complete revolution of the crankshaft, rather than time, as that is the parameter automakers use to specify the proper setting for the ignition timing in their internal combustion engines. For example, in specification sheets for timing, mechanics will find phrases like "set at 12 degrees BTDC," which translates as "set the spark plug to fire when the piston is in the position 12 degrees before top dead center of the compression stroke."

These plots would be of great interest to the mechanical design engineers responsible for selecting the metals with the necessary strength to withstand the expected speeds and accelerations of the crankshaft/connecting rod/wrist pin assembly.

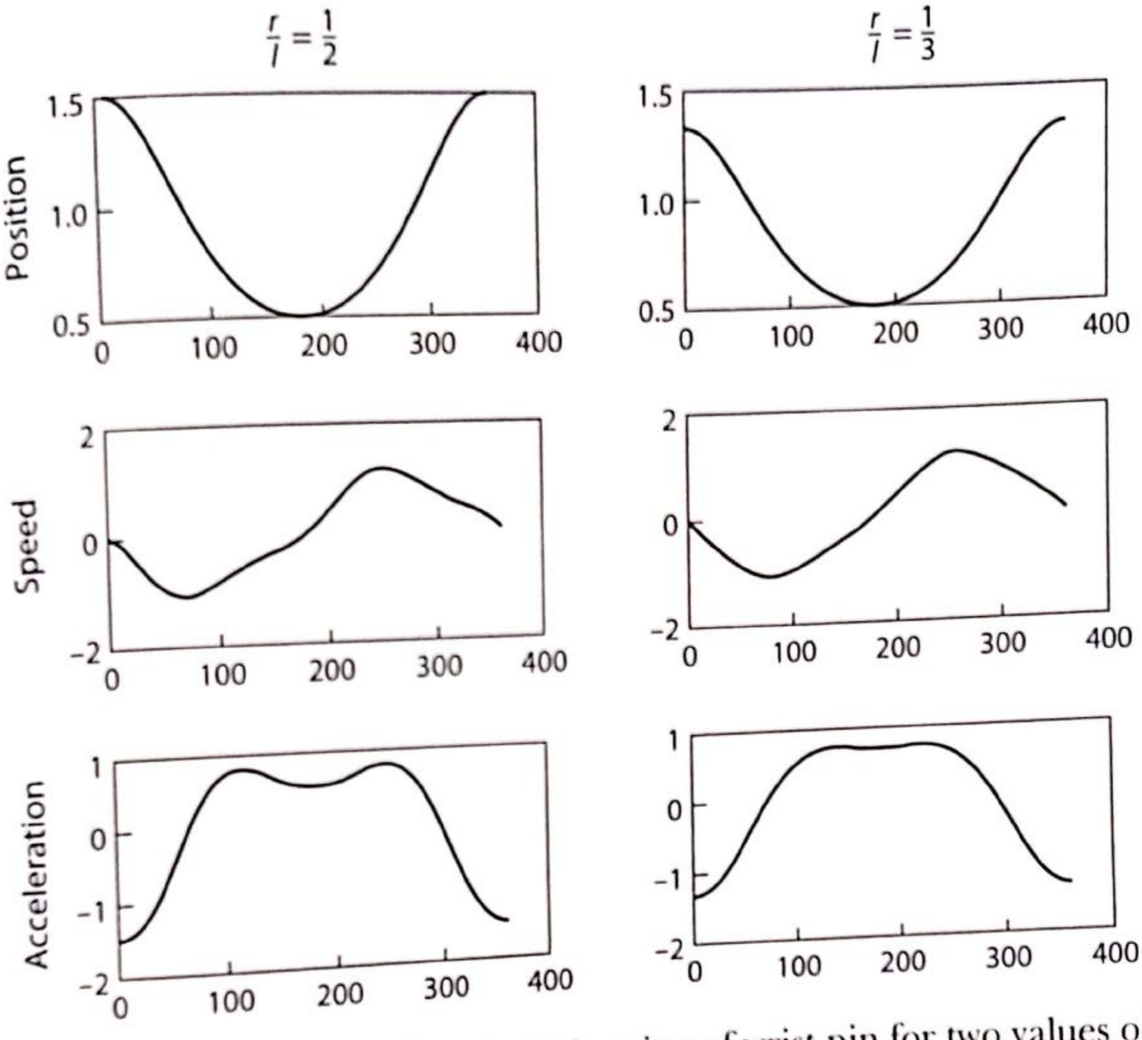

Figure 16.2. Position, speed, and acceleration of wrist pin for two values of r/l

Note

1. The units of $\omega^2 r$ are $\frac{\text{radians-squared}\cdot\text{meters}}{\text{seconds squared}}$, but *radians* are considered to be physically dimensionless.