Solution QM I 2019 Final exam

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Q1.

Eigenvalues of $\hat{J}_z$ are $\hbar$, 0, $-\hbar$.

Eigenvalues of $\hat{J}_z^2$ are $\hbar^2$, 0, $\hbar^2$.

(a)
$$\hat{H} = -\frac{D}{\hbar}\begin{pmatrix} \hbar^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar^2 \end{pmatrix}$$

$$= -D\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) If $D > 0$, ground state energy is $-D\hbar$.

If $D < 0$, ground state energy is 0.

(c) Eigenstates of $\hat{J}_z$ can be labeled as

$|1, 1\rangle$, $|1, 0\rangle$, $|1, -1\rangle$, with $j = 1$,

$m_j = 1, 0, -1$.

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Let's find $\hat{J}_+$ in the $\{ |1,1\rangle, |1,0\rangle, |1,-1\rangle \}$ basis.

$$\hat{J}_+ = \begin{pmatrix} \overset{\rightarrow 0}{\langle 1,1|\hat{J}_+|1,1\rangle} & \langle 1,1|\hat{J}_+|1,0\rangle & \overset{\rightarrow 0}{\langle 1,1|\hat{J}_+|1,-1\rangle} \\ \overset{\rightarrow 0}{\langle 1,0|\hat{J}_+|1,1\rangle} & \overset{\rightarrow 0}{\langle 1,0|\hat{J}_+|1,0\rangle} & \langle 1,0|\hat{J}_+|1,-1\rangle \\ \overset{\rightarrow 0}{\langle 1,-1|\hat{J}_+|1,1\rangle} & \overset{\rightarrow 0}{\langle 1,-1|\hat{J}_+|1,0\rangle} & \overset{\rightarrow 0}{\langle 1,-1|\hat{J}_+|1,-1\rangle} \end{pmatrix}$$

$$\hat{J}_+ |1,0\rangle = \sqrt{2}\,\hbar\,|1,1\rangle$$

$$\hat{J}_+ |1,-1\rangle = \sqrt{2-(-1)(0)}\,\hbar\,|1,0\rangle$$

$$= \sqrt{2}\,\hbar\,|1,0\rangle.$$

$$\cancel{\hat{J}_+ |1,-1\rangle = \sqrt{2}\,\hbar\,|1,}$$

$$\therefore\quad \hat{J}_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{J}_- = \hat{J}_+^{\dagger} = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

(d)

$$\hat{J}_+^{\,2} = \hbar^2 \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{J}_-^{\,2} = \hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$\hat{J}_+^{\,2} + \hat{J}_-^{\,2} = 2\hbar^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$\hat{H}_2 = -D\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\hbar A \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \hbar \begin{pmatrix} -D & 0 & 2A \\ 0 & 0 & 0 \\ 2A & 0 & -D \end{pmatrix}.$$

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Eigenstates of $\hat{H}_2$:

One of them is clearly $|1,0\rangle$.

Pick up the block $\hbar \begin{pmatrix} -D & 2A \\ 2A & -D \end{pmatrix}$.

This looks like $-D\hbar \hat{1} + \hbar 2A \hat{\sigma}_x$.

We know eigenstates of $\hat{\sigma}_x$ are $\frac{|a\rangle \pm |b\rangle}{\sqrt{2}}$. Let's

see if $\frac{|1,1\rangle + |1,-1\rangle}{\sqrt{2}} = |+\rangle$ and $\frac{|1,1\rangle - |1,-1\rangle}{\sqrt{2}} = |-\rangle$

qualify as eigenstates of this block.

Note $\frac{|1,1\rangle + |1,-1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\frac{|1,1\rangle - |1,-1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$\hbar\begin{pmatrix} -D & 2A \\ 2A & -D \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} -D+2A \\ -D+2A \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}}(-D+2A)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hurray! $\underline{\dfrac{|1,1\rangle+|1,-1\rangle}{\sqrt{2}}}$ is one eigenstate. The other one $\underline{\underline{\text{is}}}$ (must be) $\dfrac{|1,1\rangle-|1,-1\rangle}{\sqrt{2}}$.

Check:

$$\hbar\begin{pmatrix} -D & 2A \\ 2A & -D \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} -D-2A \\ 2A-D \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}}(-D-2A)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now initial state is

$$|1,-1\rangle \underset{\substack{\text{expressing as} \\ \text{eigenstates of} \\ \hat{H}_2.}}{=\!=\!=\!=} \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right).$$

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}} |1,-1\rangle$$

$$= e^{-i\frac{\hat{H}t}{\hbar}} \frac{1}{\sqrt{2}} \left(|+\rangle - |-\rangle\right)$$

$$= \frac{e^{-i(-D+2A)t}|+\rangle - e^{-i(-D-2A)t}|-\rangle}{\sqrt{2}}$$

$$= \frac{e^{iDt}e^{-i2At}|+\rangle - e^{iDt}e^{+i2At}|-\rangle}{\sqrt{2}}$$

(ignore global phase)

$$= \frac{e^{-i2At}|+\rangle - e^{+i2At}|-\rangle}{\sqrt{2}}$$

Put things back – perspective

$$= \frac{e^{-i2At}}{\sqrt{2}} \frac{1}{\sqrt{2}} \left(|1,1\rangle + |1,-1\rangle\right)$$

$$\cancel{+} - \frac{e^{+i2At}}{\sqrt{2}} \frac{1}{\sqrt{2}} \left(|1,1\rangle - |1,-1\rangle\right)$$

$$\cancel{= \frac{1}{\sqrt{2}}\left(2\cos 2At\right)|1,1\rangle}$$

$$= \frac{1}{2}\left(e^{-i2At} - e^{+i2At}\right)|1,1\rangle + \frac{1}{2}\left(e^{-i2At} + e^{+i2At}\right)|1,-1\rangle$$

$$= -\frac{1}{2}\cdot 2i \sin(2At)\,|1,1\rangle + \frac{1}{2}\cdot 2\cos(2At)\,|1,-1\rangle$$

$$= -i\sin(2At)\,|1,1\rangle + \cos(2At)\,|1,-1\rangle$$

Q2 The state $|a\rangle$ is an eigenstate of $\hat{S}_x$ with eigenvalue $\frac{3}{2}\hbar$.
Let's write this state in vector form.

$$\hat{S}_x |a\rangle = \frac{3}{2}\hbar |a\rangle$$

$$\cancel{\hbar} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \frac{3}{2}\cancel{\hbar} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

$$\sqrt{3}\, c_2 = \frac{3}{2} c_1 \Rightarrow \boxed{c_2 = \frac{\sqrt{3}}{2} c_1}$$

$$\sqrt{3}\, c_1 + 2 c_3 = \frac{3}{2} c_2$$

$$2 c_2 + \sqrt{3}\, c_4 = \frac{3}{2} c_3$$

$$\sqrt{3}\, c_3 = \frac{3}{2} c_4 \Rightarrow \boxed{c_3 = \frac{\sqrt{3}}{2} c_4}$$

$$\Rightarrow \sqrt{3} \cdot \frac{\sqrt{3}}{2}\, c_1 + 2 c_3 = \frac{3}{2} \frac{\sqrt{3}}{2} c_1$$

$$2\,C_3 = \left(3\,\frac{\sqrt{3}}{4} - \sqrt{3}\right) C_1$$

$$\boxed{C_3 = \left(\frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{2}\right) C_1}$$

$$C_4 = \frac{2}{\sqrt{3}}\left(\frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{2}\right) C_1$$

$$\boxed{C_4 = \left(\frac{3}{4} - \sqrt{3}\right) C_1}$$

$$\therefore \quad |a\rangle = N \begin{pmatrix} 1 \\ \frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{2} \\ \frac{3}{4} - \sqrt{3} \end{pmatrix}$$

N is a normalizatn constt.

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$$|\langle a|a\rangle|^2 = 1$$

$$= N^2\left(1 + \frac{3}{4} + \left(\frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{3}{4} - \sqrt{3}\right)^2\right)$$

$$= N^2\left(1 + \frac{3}{4} + \frac{27}{64} + \frac{3}{4} - \frac{9}{8} + \frac{9}{16} + 3 - \frac{3\sqrt{3}}{2}\right)$$

$$= N^2\left(\frac{64 + 48 + 27 + 48 - 72 + 36 + 192 - 96\sqrt{3}}{64}\right)$$

$$= N^2\left(\frac{115}{64} + \left(\frac{3}{4} - \sqrt{3}\right)^2\right)$$

$$= N^2\left(\frac{343 - 96\sqrt{3}}{64}\right) = 1$$

$$\Rightarrow \quad N^2 = \frac{64}{343 - 96\sqrt{3}}$$

$$N = \sqrt{\frac{64}{343 - 96\sqrt{3}}} = \underline{0.6018}$$

So x

If the input is an equal superposition of **all** eigenstates of $\underline{\underline{S_x}}$, $\frac{1}{4}$ of them will come out of the top channel.

If input state is $\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then the fraction of states coming out will be

$$\left| N \left(1 \quad \frac{\sqrt{3}}{2} \quad \frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{2} \quad \frac{3}{4} - \sqrt{3} \right) \left|\frac{3}{2}, \frac{3}{2}\right\rangle \right|^2$$

$$= N^2 (1) = (0.6018)^2 \approx 0.36.$$

36% will come out on the top channel.

Q3.

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(a) Eq.(5) suggests $\hat{H}$ has eigenstates $|1\rangle, |2\rangle$ and $|3\rangle$ with eigenvalues $\hbar\omega, 2\hbar\omega, 3\hbar\omega$ resp.

So $$|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}\left(\frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{3}}|3\rangle\right)$$

$$= \frac{1}{\sqrt{2}} e^{-i\omega t} |1\rangle - \frac{i}{\sqrt{3}} e^{-3i\omega t} |3\rangle$$

(b) $$\langle \hat{B} \rangle = \langle \psi(t)| \hat{B} |\psi(t)\rangle$$

$$= \left(\frac{1}{\sqrt{2}} e^{+i\omega t} \langle 1| + \frac{i}{\sqrt{3}} e^{+3i\omega t}\right)$$

$$b\left(2\,|1\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 2|\right)$$

$$\left(\frac{1}{\sqrt{2}} e^{-i\omega t}|1\rangle - \frac{i}{\sqrt{3}} e^{-3i\omega t}|3\rangle\right)$$

$$= b\left(\frac{1}{\sqrt{2}}\, e^{+i\omega t}\langle 1| + \frac{i}{\sqrt{3}}\, e^{+3i\omega t}\right)$$

$$\left(\frac{2}{\sqrt{2}}\, e^{-i\omega t}|1\rangle - \frac{i}{\sqrt{3}}\, e^{-3i\omega t}|2\rangle\right)$$

$$= b\left(\frac{2}{2} + \frac{1}{3}\right) = b\left(\frac{8}{6}\right)$$

$$= \frac{4b}{3}.$$

(c) At $t=0$, $|\psi(t=0)\rangle = |1\rangle$, an eigenstate of $\hat{H}$.

$$\hat{A} = a\left(|1\rangle\langle 2| + |2\rangle\langle 1| + 2|3\rangle\langle 3|\right)$$

In the eigenbasis of $\hat{H}$,

$$\hat{A} = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 2a \end{pmatrix}.$$

Eigenstates and eigenvalues of $\hat{A}$ are:

Eigenstates	Eigenvalues (measurement outcomes)
$\frac{1}{\sqrt{2}}\left(\|1\rangle + \|2\rangle\right)$	a
$\frac{1}{\sqrt{2}}\left(\|1\rangle - \|2\rangle\right)$	a
$\|3\rangle$	$2a$

Since outcome is $2a$, new state created is $|3\rangle$.

At time, t, the state will evolve into:

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|3\rangle$$

$$= e^{-3i\omega t}|3\rangle.$$

Then $\hat{B}$ is measured. We need to find outcomes and corresponding eigenstates.

$$\hat{B} = b\left(2|1\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|\right)$$

$$= \begin{pmatrix} 2b & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{pmatrix}$$

Eigenstates	Eigenvalues
$\vert 1\rangle$	$2b$
$\frac{\vert 2\rangle + \vert 3\rangle}{\sqrt{2}}$	b
$\frac{\vert 2\rangle - \vert 3\rangle}{\sqrt{2}}$	b

Hence possible outcomes are $\cancel{|2\rangle}, \cancel{|2\rangle}$, $\cancel{\frac{|2\rangle+|3\rangle}{\sqrt{2}}}$ $2b$, b, b. Let's find their probabilities given the state

$$e^{-3i\omega t}|3\rangle = |3\rangle$$ (ignore global phase).

Prob (output 2b) $= \left|\langle 1|3\rangle\right|^2 = 0.$

Prob (output b) $= \left|\left(\frac{\langle 2| + \langle 3|}{\sqrt{2}}\right)|3\rangle\right|^2$

$+ \left|\left(\frac{\langle 2| - \langle 3|}{\sqrt{2}}\right)|3\rangle\right|^2$

$= \frac{1}{2} + \frac{1}{2} = 1.$