Solution QM I 2019 Finall exam

صل

Q1.

Eigenvalues of $\hat{J}_z$ are $\hbar$, 0, $-\hbar$.

Eigenvalues of $\hat{J}_z^2$ are $\hbar^2$, 0, $\hbar^2$.

(a) $$\hat{H} = -\frac{D}{\hbar}\begin{pmatrix} \hbar^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar^2 \end{pmatrix}$$

$$= -D\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) If $D > 0$, ground state energy is $-D\hbar$.

If $D < 0$, ground state energy is 0.

(c) Eigenstates of $\hat{J}_z$ can be labeled as

$|1, 1\rangle$, $|1, 0\rangle$, $|1, -1\rangle$, with $j = 1$,

$m_j = 1, 0, -1$.

Let's find $\hat{J}_+$ in the $\{ |1,1\rangle, |1,0\rangle, |1,-1\rangle \}$ basis.

$$\hat{J}_+ = \begin{pmatrix} \overset{\to 0}{\langle 1,1|\hat{J}_+|1,1\rangle} & \langle 1,1|\hat{J}_+|1,0\rangle & \overset{\to 0}{\langle 1,1|\hat{J}_+|1,-1\rangle} \\ \overset{\to 0}{\langle 1,0|\hat{J}_+|1,1\rangle} & \overset{\to 0}{\langle 1,0|\hat{J}_+|1,0\rangle} & \langle 1,0|\hat{J}_+|1,-1\rangle \\ \overset{\to 0}{\langle 1,-1|\hat{J}_+|1,1\rangle} & \overset{\to 0}{\langle 1,-1|\hat{J}_+|1,0\rangle} & \overset{\to 0}{\langle 1,-1|\hat{J}_+|1,-1\rangle} \end{pmatrix}$$

$$\hat{J}_+ |1,0\rangle = \sqrt{2}\,\hbar\,|1,1\rangle$$

$$\hat{J}_+ |1,-1\rangle = \sqrt{2-(-1)(0)}\,\hbar\,|1,0\rangle$$

$$= \sqrt{2}\,\hbar\,|1,0\rangle.$$

$$\cancel{\hat{J}_+ |1,-1\rangle = \sqrt{2}\,\hbar\,|1,}$$

$$\therefore \quad \hat{J}_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{J}_- = \hat{J}_+^{\dagger} = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$