11. The Physics of Stacking Books

Every miser knows that a stack of pennies can be "leaned" slightly off vertical without falling. How far can the top penny be from its position in a vertical stack?"

—*Paul B. Johnson*[1]

The epigraph describes a situation that never fails to astonish all who first see it. Johnson answered his penny question by deriving a mathematical equation and solving it with some subtle arguments. Here I'll do it using just some simple physics, in which the concept of the *center of mass* of a spatially extended object will play an important role. The center of mass is the point at which we can imagine the entire mass of the object is concentrated as a *point* mass. Often, the center of mass is obvious by inspection because of symmetry. For example, the center of mass of a uniformly dense solid sphere is the geometric center of the sphere. Similarly, the center of mass of a circular hoop with uniform density is the center of the hoop (but notice there is, in this case, no mass actually at the center of mass). If the extended object is at all complicated, and symmetry arguments fail, then the center of mass has to be calculated. In the simplest case, suppose we have N point masses, m_i, $1 \leq i \leq N$, located at (x_i, y_i, z_i). Then, the x-coordinate of the center of mass is given by

$$X_C = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i},$$

and similar expressions hold for Y_C and Z_C.

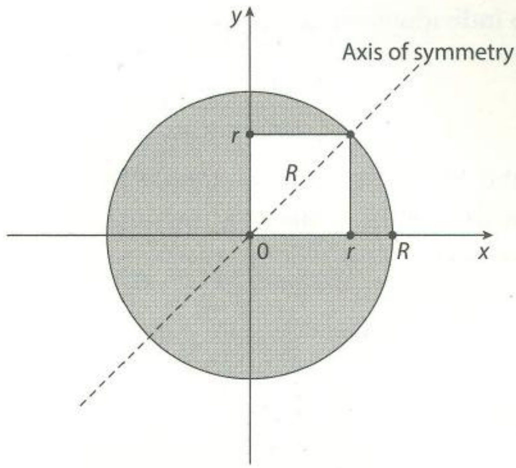

Figure 11.1. A circular disk with a square removed

Sometimes, when symmetry might *appear* to be absent, it really isn't. An example of this—a favorite of freshman physics teachers who need an exam question on short notice—is shown in Figure 11.1. There you see a circular disk of uniform thickness and density, with the largest possible square cut out of the upper-right quadrant. When the disk was still intact, symmetry told us that its center of mass was at the origin. With the square removed, however, that's no longer the case—and that's the question: where *is* the center of mass for the *cut* disk? Let's call the answer to that question (X, Y). Now, even with the cut there is still enough symmetry left in the disk to argue that $Y = X$ (that is, as the "axis of symmetry" shown in Figure 11.1 indicates, there is nothing to distinguish the x- and the y-directions). That observation helps a bit, but we are still left with the question, what is X?

The center of mass of the square cut from the disk is, by symmetry, in the middle of the square. From simple geometry (remember the Pythagorean theorem), if the radius of the disk is R, then the edge length of the square is $r = \frac{R}{\sqrt{2}}$, and so the center of the square is at $(\frac{R}{2\sqrt{2}}, \frac{R}{2\sqrt{2}})$. Now, here's the crucial observation: if we put the square back into the cut disk, we get the *intact* disk back. Who could argue with that? So, if m_1 is the mass of the cut disk and if m_2 is the mass of the square, then the formula for the center of mass resulting from combining two individual masses says

$$0 = \frac{m_1 X + m_2 \frac{R}{2\sqrt{2}}}{m_1 + m_2}.$$

The zero on the left is because, as argued by symmetry, that's the x-coordinate of the center of mass of the once-again intact disk. So, just like that, we have

$$X = -\frac{m_2}{m_1}\left(\frac{R}{2\sqrt{2}}\right).$$

Or, since the disk and the square are of uniform thickness and density, the masses of these two objects are directly proportional to their surface areas (A_1 and A_2, respectively), we can write

$$X = -\frac{A_2}{A_1}\left(\frac{R}{2\sqrt{2}}\right).$$

From geometry we have

$$A_1 = \pi R^2 - A_2,$$

and

$$A_2 = \frac{R^2}{2}.$$

So,

$$X = -\frac{\frac{R^2}{2}}{\pi R^2 - \frac{R^2}{2}}\left(\frac{R}{2\sqrt{2}}\right) = -\frac{R}{(2\pi - 1)2\sqrt{2}} = -0.06692R(= Y).$$

Isn't that slick? Okay, now that you see how the center of mass formula works, off we go to the real topics of this chapter.

Instead of Johnson's pennies (you'll see why in just a bit), imagine a book of length 1 and mass 1 lying flat on a tabletop with the book's rightmost edge right at the edge of the table, as shown in Figure 11.2. The left edge of the book is at $x = 0$, and so the right edge of the book (and the edge of the table) is at $x = 1$. The center of mass of the book is at $x = \frac{1}{2}$, and so we can slide the book forward distance $\frac{1}{2}$ before the book will fall off the table. The book projects out beyond the tabletop by $\frac{1}{2}$, and that projection is called the *overhang*, denoted by S. So, for one book, we have $S(1) = \frac{1}{2} = \frac{1}{2}(1)$.

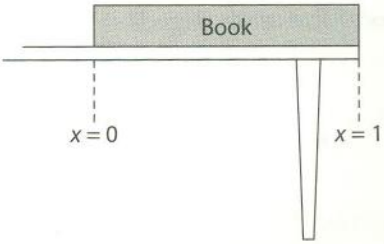

Figure 11.2. A book lying flat on a tabletop

Now, imagine two such books neatly stacked on the table. From our first analysis we know we can slide the top book forward distance $\frac{1}{2}$ before it falls off the bottom book. The center of mass of the top book is now at $x = 1$. The center of mass of the two books together is at

$$x = \frac{1\left(\frac{1}{2}\right) + 1(1)}{2} = \frac{3}{4},$$

and so we can slide the two-book combo forward distance $\frac{1}{4}$ toward the table edge before the combo falls off the table. Now, the projection of the top book beyond the table edge is

$$S(2) = \frac{1}{4} + \frac{1}{2} = \frac{1}{2}\left(1 + \frac{1}{2}\right).$$

Let's do this just one more time, with three identical books neatly stacked on the table. From our earlier results we know we can slide the top book forward by distance $\frac{1}{2}$ before it falls off the middle book, and then we can slide the upper two-book combo forward by distance $\frac{1}{4}$ before the combo falls off the bottom book. The center of mass of the upper two-book combo is now at $x = 1$. The center of mass of the three-book combo is at

$$x = \frac{1\left(\frac{1}{2}\right) + 2(1)}{3} = \frac{5}{6},$$

and so we can slide the three-book combo forward by distance $\frac{1}{6}$ toward the table edge before the combo falls off the table. Now, the projection of the top book beyond the table edge is

$$S(3) = \frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3}\right).$$

By now you have probably begun to suspect that, in general, if we keep doing this, stacking ever more books, we'll find that

$$S(n) = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}.$$

We can verify this suspicion by induction. That is, let's *suppose* that for $n-1$ books,

$$S(n-1) = \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k},$$

and then we'll show that this implies

$$S(n) = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}.$$

That would mean, since we've already shown by *direct calculation* that our supposed formula for $S(n)$ holds for $n=3$, that it must hold for $n=4$ (which means it holds for $n=5$, and so on). We also know by direct calculation that our formula holds for $n=1$ and $n=2$, as well, of course.

So, before the final adjustment of the bottom book (and all the other books above it), the top $n-1$ books have their combined center of mass at $x=1$ just before they will fall off the bottom book. The top book has a projection of $S(n-1)$ beyond the edge of the table. The center of mass of the n-book combo is at

$$x = \frac{1\left(\frac{1}{2}\right) + (n-1)(1)}{n} = \frac{1}{2n} + \frac{n-1}{n} = \frac{1+2(n-1)}{2n} = \frac{2n-1}{2n} = 1 - \frac{1}{2n}.$$

Thus, we can slide the n-book combo forward distance $\frac{1}{2n}$ toward the table edge before the n-book combo falls off the table. So,

$$S(n) = S(n-1) + \frac{1}{2n} = \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k} + \frac{1}{2n} = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k},$$

just as we suspected, and our proof by induction is done.

Now, here's the "surprise." How big can $S(n)$ be? Answer: as big as you like! That's because $S(n)$ is the truncated form of the so-called harmonic series, which is well known to blow up as $n \to \infty$.[2] As the Russian-born physicist George Gamow (1904–1968) stated in one of his books when discussing this problem:[3] "By stacking an unlimited number of books ... we can make the top book protrude any desired distance beyond the edge of the table." His very next statement, though, was far off the mark: "Because of the rapidly decreasing contribution of each new book, however, we will need the entire Library of Congress to make the overhang equal to three or four book lengths!" That is not so.

It is quite easy to program a computer to evaluate $S(n)$ for given values of n; in fact, $S(n)$ first exceeds 3 when $n = 227$, and $S(n)$ first exceeds 4 when $n = 1{,}674$. Neither value of n is anywhere near the number of books in the Library of Congress. It's an entirely different story for larger values of $S(n)$, however: the overhang $S(n)$ first exceeds 50, for example, when n is something more than 1.5×10^{43}. Now *that* is many more books than are in the Library of Congress![4]

The appearance of Paul Johnson's note on the penny-stacking problem in the *American Journal of Physics* (note 1) prompted the following reply from a physicist at The Ohio State University who had solved the problem himself some years before: "To prove [the overhang] result 'physically,' a fellow graduate student and I stacked bound volumes of *The Physical Review* one evening, until an astonishingly large offset was obtained and left them to be discovered the next morning by a startled physics librarian."[5] Who says physicists are mostly shy, quiet nerds? In my book—and as Eisner's letter demonstrates—some of them are really crazy-wild guys!

Before leaving the general topic of center of mass, I'll end this chapter by showing you a somewhat more serious application than building stacks of offset pennies and books, namely, a dramatic

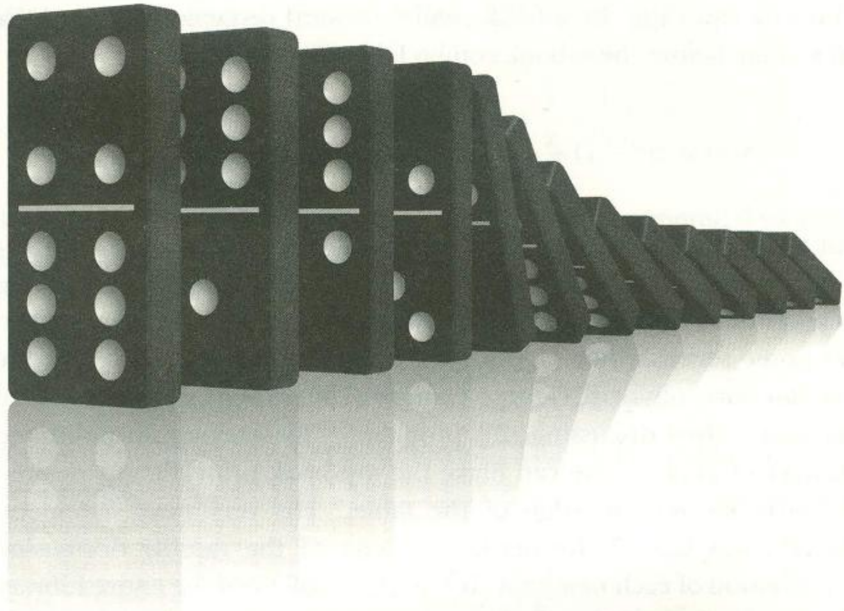

Figure 11.3. A domino chain reaction

illustration of the exponential (indeed, *explosive*) growth of energy in a chain reaction. To model neutrons successively splitting atomic nuclei, as occurs in atomic fission bombs, we'll use a falling domino to knock over a slightly larger domino, which will then knock over an even larger domino, and so on[6] (unlike the dominos in Figure 11.3, which are all the same size). The input energy required to knock over the initial domino can be quite small, while the energy released by the final falling domino can be *billions* of times larger (we'll prove this in just a bit). You can find videos of such domino chain reactions on YouTube, but they are strictly for fun viewing. Here I'll show you how to *calculate* the energies involved, using simple physics.

The communication in note 6 describes a chain of 13 ever-larger dominos, all made from acrylic plastic, with the smallest one (domino #1) having the dimensions

thickness $(w) = 1.19 \times 10^{-3}$ meters
width $(l) = 4.76 \times 10^{-3}$ meters
height $(h) = 9.53 \times 10^{-3}$ meters

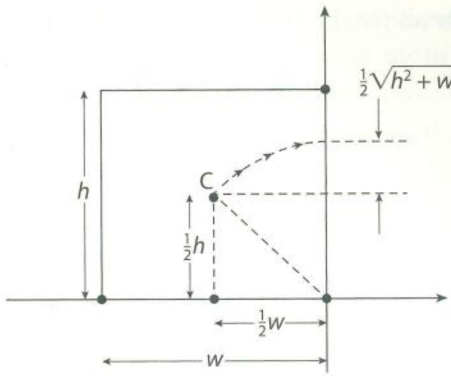

Figure 11.4. The geometry of an upright domino

and the largest one (domino #13) with the dimensions

thickness $(w) = 76.2 \times 10^{-3}$ meters
width $(l) = 305 \times 10^{-3}$ meters
height $(h) = 610 \times 10^{-3}$ meters

Starting with the smallest domino, each subsequent domino in the chain is slightly less than 1.5 times larger in each dimension than the previous one; it was stated in note 6 that the energy required to knock domino #1 over is 0.024×10^{-6} joules (see note 4 in Chapter 3 again), and the energy released by the fall of domino #13 is about 51 joules, an energy amplification factor of about 2 billion! The author of note 6 said: "It is easy to calculate [these energies]" but didn't show how to do it. So, let calculate them for ourselves.

Figure 11.4 shows a cross section of a domino, with its front face on the y-axis and its lower front edge at the origin (you are to imagine that the width, or l-dimension, is perpendicular to the page). The center of mass, C, of the domino is, by symmetry, located at the combined midpoints of each of the three dimensions. Imagine now that a force is applied to the left face of the domino. The domino will start to rotate clockwise round the lower front edge, and the center of mass will clearly rise until it is directly over the front edge. Any further rotation of the domino will place C beyond the front edge, and the domino will then topple over.

When C is directly above the front edge it will have elevated through the distance

$$\sqrt{\left(\frac{h}{2}\right)^2+\left(\frac{w}{2}\right)^2}-\frac{h}{2}=\frac{h}{2}\sqrt{1+\left(\frac{w}{h}\right)^2}-\frac{h}{2}=\frac{h}{2}\left[\sqrt{1+\left(\frac{w}{h}\right)^2}-1\right].$$

Thus, the potential energy of the domino increases by

$$\Delta E=mg\,\Delta y=mg\frac{h}{2}\left[\sqrt{1+\left(\frac{w}{h}\right)^2}-1\right],$$

where m is the mass of the domino. ΔE is the required input energy to topple the domino. The mass m is

$$m=\rho wlh,$$

where ρ is the density of acrylic plastic. A quick search on the Web gave the value of ρ as somewhere between 1.15 and 1.2 grams/cubic centimeter; I'll use an average of 1.18 grams/cubic centimeter $= 1.18\times 10^3\,\frac{\text{kilograms}}{\text{cubic meter}}$. So, for domino #1, the mass is

$$\begin{aligned}m &= 1.19\times 4.76\times 9.53\times 10^{-9}\text{ cubic meters}\times 1.18\times 10^3\,\frac{\text{kilograms}}{\text{cubic meter}}\\ &= 63.7\times 10^{-6}\text{ kilograms,}\end{aligned}$$

and therefore

$$\begin{aligned}\Delta E &= \frac{1}{2}63.7\times 10^{-6}\text{ kilograms}\times 9.8\,\frac{\text{meters}}{\text{seconds squared}}\\ &\quad\times 9.53\times 10^{-3}\text{ meters}\left[\sqrt{1+\left(\frac{1.19\times 10^{-3}}{9.53\times 10^{-3}}\right)^2}-1\right]\\ &= 2,975\times 10^{-9}\,\frac{\text{kilograms}\cdot\text{meters-squared}}{\text{seconds squared}}(0.00777)\\ &= 23\times 10^{-9}\text{ joules}\\ &= 0.023\times 10^{-6}\text{ joules,}\end{aligned}$$

which is very close to the value declared by the author of note 6 (who suggested that this really quite small energy input could "be supplied by nudging [the domino] with a long wispy piece of cotton batton.")

Finally, to compute the energy released by the toppling of the largest domino, we start with its initial energy and then add the energy required to raise its center of mass to the point where it is over the domino's front edge. We then subtract the potential energy retained by the domino after it has fallen over. The result is the energy released by the domino. So, when domino #13 is upright its center of mass is at height 305×10^{-3} meters. When it's hit by domino #12 the center of mass of domino #13 rises to a height of

$$\frac{1}{2}\sqrt{(610)^2 + (76.2)^2} \times 10^{-3} \text{ meters} = 307.4 \times 10^{-3} \text{ meters.}$$

When domino #13 has toppled, the original w dimension is the new h dimension, and so the center of mass is at height 38.1×10^{-3} meters. The change (decrease) in the potential energy of the domino is therefore

$$\begin{aligned} mg\Delta y &= \rho wlhg\Delta y \\ &= 1.18 \times 10^3 \frac{\text{kilograms}}{\text{cubic meter}} \times 9.8 \frac{\text{meters}}{\text{seconds squared}} \\ &\quad \times 305 \times 76.2 \times 610 \times 10^{-9} \text{ cubic meters} \\ &\quad \times (307.4 - 38.1) \times 10^{-3} \text{ meters} = 44 \text{ joules.} \end{aligned}$$

This result is "close" to 51 joules but still far enough off to warrant some concern. My guess is that the author of note 6 simply did a rough calculation and ignored the fact that the toppled center of mass was actually not at zero height. That is, he did the $mg\Delta y$ calculation but used 307.4×10^{-3} meters for Δy, which would result in a potential energy decrease of 50.4 joules.

The energy amplification factor achieved by the 13 falling dominos is, by the calculations here, the quite impressive value of

$$\frac{44}{0.023 \times 10^{-6}} = 1.9 \times 10^9 = 1.9\,billion!$$

Notes

1. These are the opening words in Johnson's cleverly titled note that simultaneously alludes to Italian money and that country's famous tower in Pisa: "Leaning Tower of Lire," *American Journal of Physics*, April 1955, p. 240.

2. Here's a simple demonstration of that:

$$\begin{aligned}\lim_{n\to\infty} S(n) &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\dots\\ &> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\dots\\ &> 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots,\end{aligned}$$

where we continuously replace each new subsequence of terms with length 2^k (where $k \geq 1$) in the original series with a smaller subsequence that sums to $\frac{1}{2}$. Thus, a *lower* bound on the sum, is infinity, and so $\lim_{n\to\infty} S(n) = \infty$.

3. George Gamow, *Matter, Earth, and Sky* (2nd ed.), Prentice-Hall, 1965, p. 20. Gamow didn't actually derive $S(n)$, as done here, but simply alluded to it.

4. This huge numerical value (it's *far* bigger than the number of stars in the Universe, estimated to be a "mere" 10^{22}) obviously can't be found by simply running a computer summation of the harmonic series. For an explanation of how it was computed, see R. P. Boas, Jr, and J. W. Wrench, Jr, "Partial Sums of the Harmonic Series," *American Mathematical Monthly*, October 1971, pp. 864–870, which gives the exact value of n for which $S(n)$ first exceeds 50: $n = 15092688622113788323693563264538101449859498$. Do you know how to even *say that*? I don't!

5. Leonard Eisner, "Leaning Tower of *the Physical Reviews*," *American Journal of Physics*, February 1959, pp. 121–122.

6. This discussion on dominos is inspired by a brief note written by Lorne A. Whitehead, "Domino 'chain reaction,'" *American Journal of Physics*, February 1983, p. 182.