# Assignment 5 : Solution

1. The density of states for spin ↑ (↓) is given by $g_\uparrow(E) = g_\downarrow(E) = \frac{1}{2}CE^{\frac{1}{2}}$. Draw a sequence of diagrams showing spin-split bands at 0 K with

   (a) Zero applied magnetic field.

   (b) Applied field $B$ with the density of states shifting before equilibrium is attained.

   (c) The achievement of equilibrium.
   Calculate the magnetization $M$ and susceptibility $\chi$ using the relationship

$$\chi = \frac{M}{H} = \frac{\mu_o M}{B}. \tag{1}$$

**Solution:**

From Figure (c),

$$\begin{aligned}
M &= \mu_B(n_\uparrow - n_\downarrow) \\
&= \mu_B\left[\int g_\uparrow(E_F + \mu_B B)dE - \int g_\downarrow(E_F - \mu_B B)dE\right] \\
&= \mu_B\left[\int \frac{1}{2}g(E_F + \mu_B B)dE - \int \frac{1}{2}g(E_F - \mu_B B)dE\right] \\
&= \frac{\mu_B}{2}\left[\int g(E_F + \mu_B B)dE - \int g(E_F - \mu_B B)dE\right]
\end{aligned}$$

Using Taylor Expansion $\Delta = 2\mu_B B$

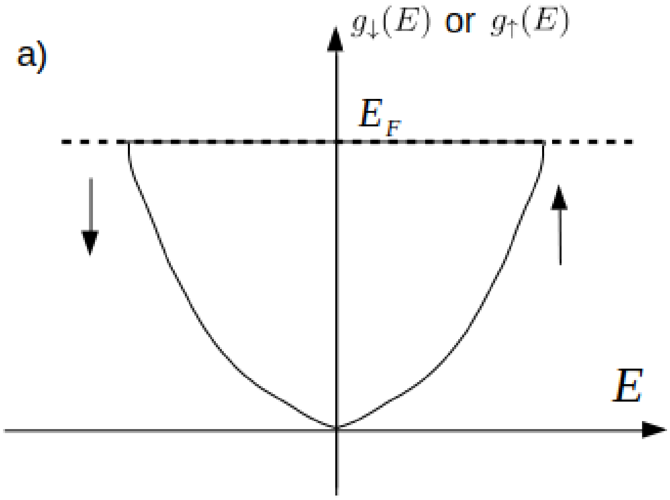
$$\begin{aligned}
M &= \frac{2(\mu_B)^2 B}{2}\int dE \frac{dg(E)}{dE}\Big|_{E_F} \\
&= \mu_B^2 B g(E_F) \\
&= \mu_B^2 B C E_F^{\frac{1}{2}}
\end{aligned}$$

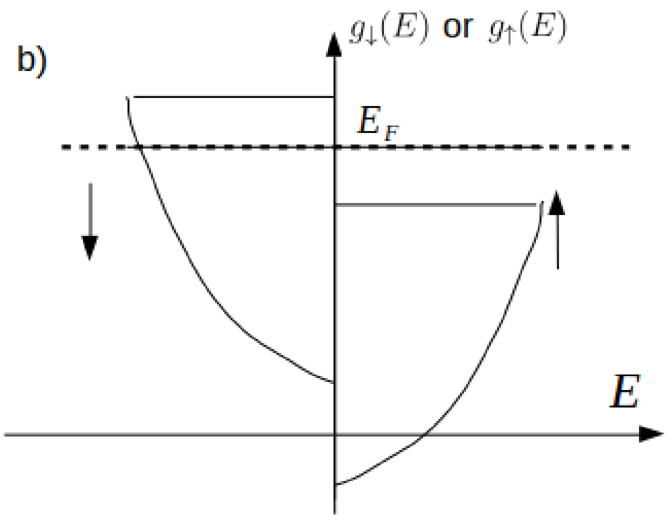
Susceptibility becomes,

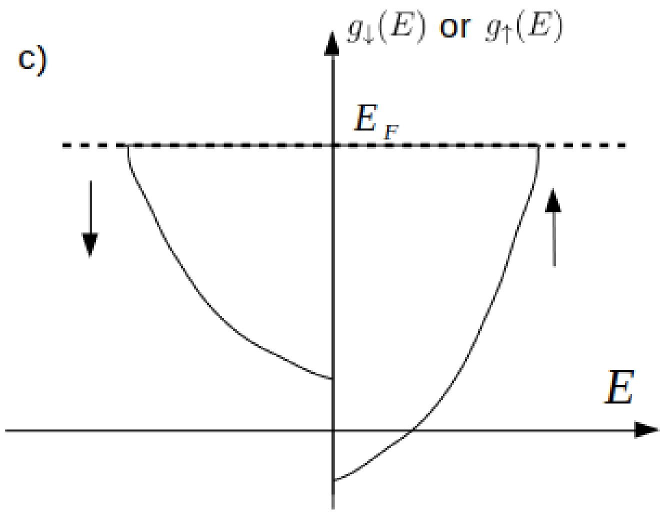
$$\begin{aligned}
\chi &= \frac{\mu_o}{B} M \\
\chi &= \mu_o \mu_B^2 C E_F^{\frac{1}{2}}
\end{aligned} \tag{2}$$

For C, we have

$$\begin{aligned}
n &= \int_0^{E_F} C E^{\frac{1}{2}} dE \\
&= \frac{2C}{3} E_F^{\frac{3}{2}}.
\end{aligned}$$


a)
$g_\downarrow(E)$ or $g_\uparrow(E)$
$E_F$
$E$


b)
$g_\downarrow(E)$ or $g_\uparrow(E)$
$E_F$
$E$


c)
$g_\downarrow(E)$ or $g_\uparrow(E)$
$E_F$
$E$

Which gives,

$$C = \frac{3n}{2}\frac{1}{E_F^{\frac{3}{2}}}$$

Put in equation (2),

$$\begin{aligned}\chi &= \mu_o \mu_B^2 \frac{3n}{2}\frac{E_F^{\frac{1}{2}}}{E_F^{\frac{3}{2}}} \\ &= \mu_o \mu_B^2 \frac{3n}{2}\frac{1}{E_F}\end{aligned}$$

2. Find the density of states for quantized free electrons in a one dimensional and two dimensional solid. For two dimensions, show that the chemical potential ($\mu$) is independent of $T$ provided $T \ll \mu$.
   **Solution:**

   (a) **Density of states in 2D:**
   In ground state of a system of N free electrons, the occupied orbitals may be represented as points inside a sphere in **k** space. For 2D, we will have circle instead of sphere. The energy at the circumference of the circle is the Fermi energy; the wavevectors at the Fermi surface have a magnitude $k_F$ such that:

   $$E_F = \frac{\hbar^2}{2m}k_F^2 \tag{3}$$

   Area of the circle is,

   $$V_F = \pi k_F^2 \tag{4}$$

   Area of each point is $(\frac{2\pi}{V})^2$. Total number of points is

   $$N = 2.\frac{\pi k_F^2}{(2\pi/L)^2} = \frac{k_F^2 L^2}{2\pi}$$

   where the factor 2 on the left comes from the allowed values of the spin quantum number for each allowed value of **k**. Then we gets

   $$k_F = \left(\frac{2\pi^2 N}{L^2}\right)^{\frac{1}{2}} \tag{5}$$

   which depends only on the particle concentration.
   Density of states,

   $$g(E) = \frac{dN}{dE} \tag{6}$$

Total number of points inside a circle of radius $k$,

$$N(k) = \frac{k^2L^2}{2\pi}$$
$$\frac{dN}{dk} = \frac{2kL^2}{2\pi} = \frac{kL^2}{\pi}$$

Using equation (3),

$$\frac{dE}{dk} = \frac{\hbar^2 k}{m} \tag{7}$$
$$g(E) = \frac{dN}{dE} = \frac{dN}{dk}\frac{dk}{dE}$$
$$= \left(\frac{kL^2}{\pi}\right)\left(\frac{m}{k\hbar^2}\right)$$
$$g(E) = \frac{L^2 m}{\pi\hbar^2} \tag{8}$$

(b) Density of states in 1D:
For 1D, we will have a Fermi line in k space. The Fermi energy:

$$E_F = \frac{\hbar^2}{2m}k_F^2 \tag{9}$$

Length of the line is,

$$V_F = \pi k_F^2 \tag{10}$$

Length of each point is $\frac{2\pi}{V}$. Total number of points is

$$N = 2.\frac{k_F}{(2\pi/L)} = \frac{k_F L}{\pi}$$

where the factor 2 on the left comes from the allowed values of the spin quantum number for each allowed value of $\mathbf{k}$. Then we gets

$$k_F = \left(\frac{\pi N}{L}\right) \tag{11}$$

which depends only on the particle concentration.
Density of states,

$$g(E) = \frac{dN}{dE} \tag{12}$$

Total number of points on the line of length $k$,

$$N(k) = \frac{kL}{\pi}$$
$$\frac{dN}{dk} = \frac{L}{\pi}$$

Using equation (3),

$$\frac{dE}{dk} = \frac{\hbar^2 k}{m} \tag{13}$$

$$\begin{aligned} g(E) &= \frac{dN}{dE} = \frac{dN}{dk}\frac{dk}{dE} \\ &= \left(\frac{L}{\pi}\right)\left(\frac{m}{k\hbar^2}\right) \\ &= \left(\frac{L}{\pi}\right)\left(\frac{m}{\left(\frac{2mE}{\hbar^2}\right)^{(1/2)}\hbar^2}\right) \end{aligned}$$

$$g(E) = \frac{Lm^{1/2}}{\pi\hbar\sqrt{2}}E^{(-1/2)} \tag{14}$$

**Chemical Potential for density of states in 2D:**
At $T > 0K$, number of point per unit area,

$$n = \int_0^\infty dE f(E) g(E) \tag{15}$$

Using Sommerfeld expansion,

$$n = \int_0^\infty dE f(E) \frac{d\Gamma(E)}{dE} = \Gamma(E)|_\mu + \frac{\pi^2}{6}(k_B T)^2 \frac{d^2\Gamma(E)}{dE^2} \tag{16}$$

where,

$$\Gamma(E) = \int_0^E g(E')dE'$$

$$\Gamma(E)|_\mu = \int_0^\mu g(E)dE$$

Density of states in 2D:

$$\begin{aligned} g(E) &= \frac{L^2 m}{\pi\hbar^2} = C \\ \Gamma(E)|_\mu &= \int_0^\mu C dE = CE|_\mu \end{aligned}$$

Equation (16) becomes,

$$\begin{aligned} n &= CE|_\mu + \frac{\pi^2}{6}(k_B T)^2 \frac{dg(E)}{dE} \\ &= CE|_\mu + \frac{\pi^2}{6}(k_B T)^2 (0) \\ n &= C\mu \end{aligned} \tag{17}$$

Assuming number of point don't change with temperature so we can write,

$$n = \int_0^{E_F} dEg(E) = CE_F \tag{18}$$

Comparing equation (17) and (18), w get

$$\mu = E_F \tag{19}$$

Clearly, chemical potential is independent of temperature (T).

3. Describe how the internal energy $U(T)$ depends on temerature $T > 0$. Use the Sommerfeld expansion. Derive the specific heat capacity $C_V = \frac{\partial u}{\partial T}|_V$ from the expression deduced.
**Solution:**
Internal energy per unit volume of the solid:

$$u = \int_0^{\infty} dEf(E)Eg(E) \tag{20}$$

Using Sommerfeld expansion,

$$u = \int_0^{\infty} dEf(E)\frac{d\Gamma(E)}{dE} = \Gamma(E)|_\mu + \frac{\pi^2}{6}(k_BT)^2\frac{d^2\Gamma(E)}{dE^2} \tag{21}$$

where,

$$\Gamma(E) = \int_0^{E} g(E')EdE'$$

$$\Gamma(E)|_\mu = \int_0^{\mu} g(E)EdE$$

Similarly,

$$\Gamma''(E)|_\mu = \frac{d^2}{dE^2}\int_0^{\mu} g(E)EdE = \frac{d^2}{dE^2}(Eg(E)) = g(E) + Eg'(E)$$

Density of states in 2D:

$$g(E) = \frac{L^2m}{\pi\hbar^2} = C$$

$$\Gamma(E)|_\mu = \int_0^{\mu} CdE = CE|_\mu$$

Density of states:

$$g(E) = \frac{m}{\pi\hbar^2}\frac{(2mE)^{\frac{1}{2}}}{\hbar} = CE^{\frac{1}{2}}$$

$$\Gamma(E)|_\mu = \int_0^{\mu} ECE^{\frac{1}{2}}dE = \int_0^{\mu} CE^{\frac{3}{2}}dE = \frac{2}{5}CE^{\frac{5}{2}}|_\mu$$

Equation (21) becomes,

$$
\begin{aligned}
u &= \frac{2}{5}CE^{\frac{5}{2}}|_{\mu} + \frac{\pi^2}{6}(k_BT)^2(CE^{\frac{1}{2}} + E(\frac{1}{2}CE^{-\frac{1}{2}}))|_{\mu} \\
&= \frac{2}{5}CE^{\frac{5}{2}}|_{\mu} + \frac{\pi^2}{6}(k_BT)^2((3/2)CE^{\frac{1}{2}})|_{\mu} \\
&= \frac{2}{5}C\mu^{\frac{5}{2}} + \frac{\pi^2}{6}(k_BT)^2((3/2)C\mu^{\frac{1}{2}})
\end{aligned} \tag{22}
$$

Chemical potential:

$$\mu = E_F\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_{E_F}}\right)^2\right) \tag{23}$$

Putting in equation (22),

$$u = \frac{2}{5}CE_F^{\frac{5}{2}}\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_{E_F}}\right)^2\right)^{\frac{5}{2}} + \frac{\pi^2}{6}(k_BT)^2((3/2)CE_F^{\frac{1}{2}})\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_{E_F}}\right)^2\right)^{\frac{1}{2}}$$

As $T_{EF} >> T$, we can use binomial expansion,

$$u = \frac{2}{5}CE_F^{\frac{5}{2}}\left(1 - \frac{5}{2}\frac{\pi^2}{12}\left(\frac{T}{T_{E_F}}\right)^2 + ....\right) + \frac{\pi^2}{6}(k_BT)^2((3/2)CE_F^{\frac{1}{2}})\left(1 - \frac{1}{2}\frac{\pi^2}{12}\left(\frac{T}{T_{E_F}}\right)^2 + ....\right)$$

Ignoring the higher terms,

$$
\begin{aligned}
u &= \frac{2}{5}CE_F^{\frac{5}{2}}\left[1 - \frac{5}{2}\frac{\pi^2}{12}\left(\frac{T}{T_{E_F}}\right)^2 + \frac{5}{2}\frac{\pi^2}{6}\frac{3}{2}(k_BT)^2\left(\frac{1}{E_F^2}\right) - \frac{5}{2}\frac{\pi^2}{6}(k_BT)^2\frac{1}{2}\frac{\pi^2}{12}\left(\frac{T}{T_{E_F}}\right)^2\frac{1}{E_F^2}\right] \\
&= u_o\left[1 - \frac{5}{2}\frac{\pi^2}{12}\left(\frac{T}{T_{E_F}}[-1+3] + O(4)\right)\right]
\end{aligned}
$$

Finally, we get

$$u = u_o\left[1 + \frac{5}{12}\pi^2\left(\frac{T}{T_{E_F}}\right)\right] \tag{24}$$

Specific heat capacity,

$$C_v = \frac{du}{dT} = \frac{5u_o}{12}\pi^2\left(\frac{2T}{T_{E_F}^2}\right) = \frac{5\pi^2 u_o}{6T_{E_F}^2}T \tag{25}$$