

General Theory of Heart-Vector Projection

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A mathematical and physical basis for heart-vector projection concepts is presented and its significance for current practices in frontal-plane and spatial electrocardiography is discussed. Assumptions underlying the theory are stated and utilized to derive a general relationship between the heart dipole and the potentials it produces in the body. Specific equations are given for unipolar and bipolar leads and the Wilson central-terminal voltage. Coefficients determined experimentally for human torso models differ markedly from those that stem from Einthoven's hypothesis. Einthoven's equilateral triangle represents a special case of the general theory. Geometric interpretations are applied to a quantitative frontal-plane example, to the Wilson central-terminal voltage and extended to the concept of an image surface corresponding to the human body surface.

HEART-VECTOR projection concepts applicable to volume conductors other than a homogeneous sphere are being employed by a gradually increasing number of workers in electrocardiographic research.¹⁻⁶ These improved concepts represent a significant advance in electrocardiographic theory which should enable a far more accurate analysis of human-body waveforms produced by the heart beat than has been available heretofore.

Einthoven's "schema," introduced in 1913, heralded the beginning of heart-vector projection ideas.⁷ This work led to thinking in terms of the manifest heart vector. Numerous aids to the application of this method of electrocardiographic analysis appeared, such as Dieuade's chart⁸ and Mann's monocardio-gram.⁹ In 1927, Canfield presented an analysis of a centric dipole (doublet) in a homogeneous conducting sphere in which the boundary potential was shown to be a cosine function of angular electrode position.¹⁰ This cosine law also applies in cases of the infinite plane lamina,¹¹ the circular disc with a centric dipole,¹¹ and the triangular lamina with a centric dipole,¹² and is the basis for simplified vector-projection methods. These vector-projection

ideas became deeply entrenched as one interpretation followed another; they are still being held tenaciously by most electrocardiographers. Moreover, most investigators in the field of spatial vectorcardiography have proceeded to apply this same cosine law for the various three-dimensional systems of electrodes involved: tetrahedron, cube, rectangle, etc. Vector projections of the manifest heart vector on the sides of an equilateral triangle (in the case of frontal-plane electrocardiography) and projections on a line joining other electrodes on the human subject (in the case of vectorcardiography) are oversimplifications and entail sizable errors.^{1, 2, 6} As will be shown they are special cases of a general theory of heart-vector projection.

Burger and van Milaan^{1, 2} extended the concepts of heart-vector projection and applied these broadened ideas in their electrocardiographic research. They and others^{3-6, 13} have shown that substantial errors can arise when waveforms produced by the heart beat are interpreted in terms of the simplified vector-projection theory that involves a centric dipole in a homogeneous conducting sphere.

In essence, the hypothesis that the human electrical system can be represented by a centric dipole in a homogeneous conducting medium lies at the root of the simple vector-projection ideas which are accurately applicable only for such an idealized system and not for the human subject.^{1, 6} These vector-projection ideas have been found convenient as an aid in interpretation by focusing atten-

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tion on heart-vector characteristics rather than on detailed waveshapes of body-surface voltages. Moreover, the simple rules of vector projection have been readily applied. Refinement of the theoretic treatment of body-surface voltages, however, must keep pace with the insight and information obtained over the past 40 years concerning the nature of the body-surface potential differences arising from the heart beat.

The present theoretic treatment is, essentially, a reiteration, expansion and formalization of the mathematical and physical basis for heart-vector projection ideas initiated by Einthoven,⁷ embellished by many other workers, and extended recently by the fine work of Burger and van Milaan.^{1, 2} It is hoped that the present treatment will clarify vector relationships which are applicable and that it will stimulate use of more accurate vector-projection analysis than is presently employed in connection with the human subject. The use of more refined analysis for the frontal-plane electrocardiogram well might lead to an improved understanding of heart action. But especially the application of the theoretic concepts presented may help to channel the efforts of workers in the field of spatial vector-cardiography along more fruitful lines with a reduction in efforts devoted to the use of inapplicable methods of analysis.

ASSUMPTIONS

Three assumptions are required to establish the general theory of heart-vector projection. They are: (1) that the human body is a heterogeneous linear resistive electrical conducting medium, (2) that the distribution of currents associated with electrical activation of heart muscle may be represented at each instant of time during the cardiac cycle by a single equivalent current dipole whose orientation (axis direction) and moment are variable and functions of the actual current distribution, and (3) that the equivalent heart dipole remains fixed in position during the cardiac cycle.

These assumptions, represented schematically in figure 1, are far less restrictive than those which underlie currently used vector-

projection methods as can be seen from the following: (a) The medium boundary can have any shape; it need not be a sphere. (b) The electrode positions are entirely unrestricted; they need not be at the apices of an equilateral triangle. (c) The medium can contain any inhomogeneities in any conceivable configuration; a homogeneous medium is not required. (d) The dipole may be located at any point in the medium; it need not be centrally located. It is assumed, however, that its position is fixed.

Although applicability of these assumptions has not been firmly established, the theory about to be developed is much less restricted than that of a centric dipole in a homogeneous conducting sphere. It is believed that the linear medium assumption will stand indefinitely. While the equivalent heart dipole assumption may not be applicable during all instants of the cardiac cycle^{1, 14, 15} its abandonment would necessitate discarding the vector theory almost entirely. The assumption of a fixed dipole position is probably suitable for present-day work, but as this factor becomes

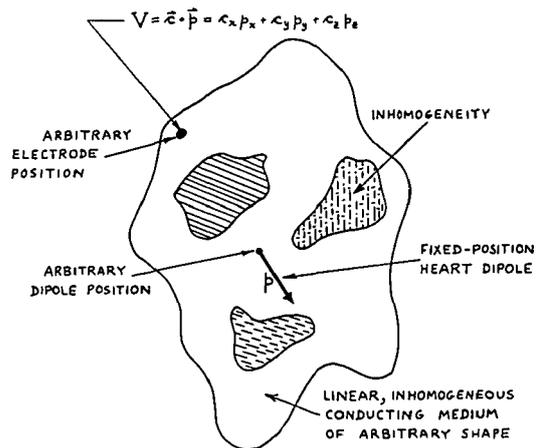


FIG. 1. Schematic illustration of a general linear three-dimensional, heterogeneous volume conductor of any shape. A current dipole of variable moment p is fixed in position at any arbitrary point. Potential V at any point inside or on the boundary is given by the dot product of the dipole vector p and a vector c whose components c_x , c_y and c_z are numbers which depend upon the medium size, shape, conductivity and distribution of inhomogeneities as well as on dipole position and location of the pickup electrode. Numerical components do not depend on p .

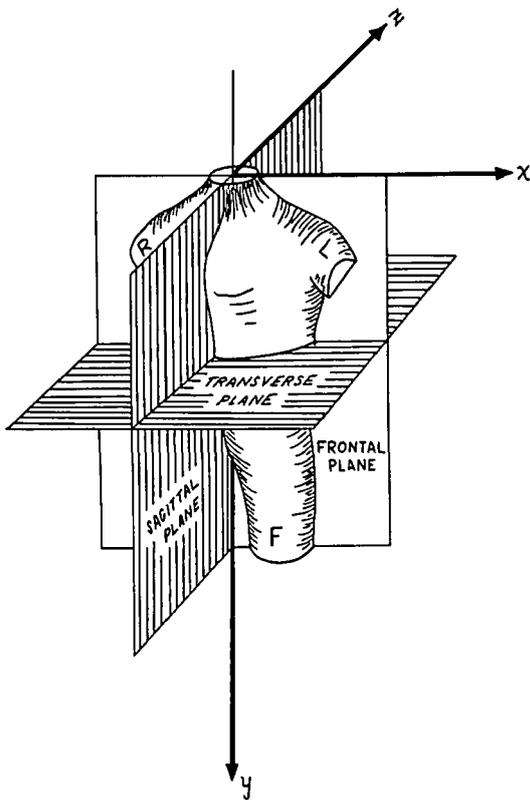


FIG. 2. Sketch of the human torso showing rectangular coordinate system with x , y and z axes adopted tentatively as standard for vectorcardiography. (Discussion in text.)

better understood the theory may have to be modified to take the temporal changes of dipole location^{5, 16} into account.

GENERAL THEORY

With the foregoing three assumptions, it is possible to formulate a general relationship between the heart dipole moment and the potential it produces at any point in or on the boundary of the medium. Let the heart dipole be represented by a vector*

$$\mathbf{p} = i p_x + j p_y + k p_z$$

where p_x , p_y and p_z are the three scalar components of the dipole moment and are functions of time, and i , j and k are standard unit vectors¹⁷ of a rectangular coordinate system. The potential difference V between any point

* All vectors are set in bold-face type throughout this paper.

in the medium and the midpotential of the dipole, assigned the value zero arbitrarily, can be expressed as

$$V = \mathbf{c} \cdot \mathbf{p} = c_x p_x + c_y p_y + c_z p_z \quad (1)$$

as a direct consequence of the linearity of the medium, where the vector $\mathbf{c} = i c_x + j c_y + k c_z$ is a function of the shape, size, and characteristics of the medium, the position of the dipole and the location of the point in the medium where the potential is V . Equation 1 is essentially a statement of the law of superposition which is valid for any linear medium. The components of the vector \mathbf{c} are real functions, since the medium is assumed to be resistive, which become numerical constants if the factors cited above upon which they depend are fixed. Thus, for a given subject with a pickup electrode fixed in a given position, the components c_x , c_y and c_z are constant, accepting the assumptions which have been stated.

From this single equation stems the entire general theory of heart-vector projection which includes as a special case the theory of a centric dipole in a homogeneous conducting sphere.

ELECTRODE VOLTAGES

Body-surface and internal voltages of interest in electrocardiography can all be expressed in the linear form of equation 1. For definiteness, several of the more commonly used voltages will be given explicitly in terms of the rectangular coordinate system shown in figure 2.

Unipolar Voltages. Unipolar voltages are defined theoretically as potential differences measured with respect to the dipole midpotential, and they are expressible directly in the form of equation 1. For the limb electrodes the unipolar voltages are

$$\begin{aligned} V_R &= \mathbf{c}_1 \cdot \mathbf{p} = c_{1x} p_x + c_{1y} p_y + c_{1z} p_z \\ V_L &= \mathbf{c}_2 \cdot \mathbf{p} = c_{2x} p_x + c_{2y} p_y + c_{2z} p_z \\ V_F &= \mathbf{c}_3 \cdot \mathbf{p} = c_{3x} p_x + c_{3y} p_y + c_{3z} p_z \end{aligned} \quad (2)$$

where the subscripts R , L and F signify the right arm, left arm and left leg (foot) respectively, and the subscripts 1, 2 and 3 are associated with R , L and F , respectively. The

components of the unipolar vectors \mathbf{c}_1 , \mathbf{c}_2 and \mathbf{c}_3 are numerical constants for a given human subject. These constants, nine in all (three for each electrode), are independent in the general case; therefore, the sum $V_R + V_L + V_F$ is not generally zero, from which it follows that the Wilson central-terminal voltage is not zero, as discussed later.*

For any other electrode, the unipolar voltage may be expressed in this same form; the magnitude and direction of the constant unipolar vector which is applicable to the particular electrode in question must be used. For example, the unipolar voltage at the electrode on the back of the human subject, V_B , is given by

$$V_B = \mathbf{c}_4 \cdot \mathbf{p} = c_{4x}p_x + c_{4y}p_y + c_{4z}p_z \quad (3)$$

Similar relations apply for chest electrodes, shoulder and hip electrodes and even for electrodes that might be inserted in the human body.

Bipolar Leads. The expressions for the limb-lead voltages can be obtained from their definitions in terms of unipolar voltages:

$$\begin{aligned} V_I &= V_L - V_R = \mathbf{A} \cdot \mathbf{p} \\ &= A_x p_x + A_y p_y + A_z p_z \\ V_{II} &= V_F - V_R = \mathbf{B} \cdot \mathbf{p} \quad (4) \\ &= B_x p_x + B_y p_y + B_z p_z \\ V_{III} &= V_F - V_L = \mathbf{C} \cdot \mathbf{p} \\ &= C_x p_x + C_y p_y + C_z p_z \end{aligned}$$

where the limb-lead vectors \mathbf{A} , \mathbf{B} and \mathbf{C} are defined in terms of the unipolar vectors as follows: $\mathbf{A} = \mathbf{c}_2 - \mathbf{c}_1$, $\mathbf{B} = \mathbf{c}_3 - \mathbf{c}_1$ and $\mathbf{C} = \mathbf{c}_3 - \mathbf{c}_2$. The relation $V_I + V_{III} = V_{II}$ is, of course, satisfied for all values of p_x , p_y and p_z which are functions of time, and the vector relation $\mathbf{A} - \mathbf{B} + \mathbf{C} = 0$ follows directly from their definitions.

Similarly, any other bipolar lead can be expressed in terms of unipolar voltages at

* This implies, of course, that true unipolar voltages are not measured in current electrocardiographic practice; instead these so-called unipolar voltages are potential differences between the electrode on the body and the Wilson central terminal. The sum of these apparent R , L and F unipolar voltages is inevitably zero.¹³

each of the two electrodes; for example, a bipolar lead from an electrode on the back to the right arm (defining a potential rise from the back to the arm as a positive quantity) can be written

$$\begin{aligned} V_{BR} &= V_B - V_R = \mathbf{D} \cdot \mathbf{p} \\ &= D_x p_x + D_y p_y + D_z p_z \end{aligned} \quad (5)$$

where $\mathbf{D} = \mathbf{c}_4 - \mathbf{c}_1$.

Solution for Heart Vector. A major objective in electrocardiography is to determine the heart-dipole variations during the cardiac cycle. Since the heart dipole has three independent components, it is necessary to measure three independent potential differences simultaneously in order to determine these components, and this demands the use of at least four different electrode positions. As an illustration, suppose the three limb electrodes are used with a fourth electrode on the back, an arrangement used by some vectorcardiographers.¹⁹ Then it is possible to derive explicit equations for p_x , p_y and p_z in terms of the coefficients and the electrode potential differences. Solving V_I and V_{II} of equation 4 and V_{BR} of equation 5 simultaneously for p_x , p_y and p_z results in

$$\begin{aligned} p_x &= \frac{1}{\Delta} [V_I(B_y D_z - B_z D_y) \\ &\quad + V_{II}(A_z D_y - A_y D_z) \\ &\quad + V_{BR}(A_y B_z - A_z B_y)] \\ p_y &= \frac{1}{\Delta} [V_I(B_z D_x - B_x D_z) \\ &\quad + V_{II}(A_x D_z - A_z D_x) \\ &\quad + V_{BR}(A_x B_x - A_x B_z)] \quad (6) \\ p_z &= \frac{1}{\Delta} [V_I(B_x D_y - B_y D_x) \\ &\quad + V_{II}(A_y D_x - A_x D_y) \\ &\quad + V_{BR}(A_x B_y - A_y B_x)] \end{aligned}$$

where Δ is the determinant of the set of three equations:

$$\begin{aligned} \Delta &= A_x(B_y D_z - B_z D_y) + A_y(B_z D_x - B_x D_z) \\ &\quad + A_z(B_x D_y - B_y D_x) \end{aligned}$$

When the components of the vectors A , B and D are known as numbers these equations become far less formidable since Δ then becomes a single number as do the multiplying coefficients of V_I , V_{II} and V_{BR} .

Wilson Central-Terminal Voltage. The Wilson central terminal is formed by connecting three equal resistors from R , L and F to a common junction. The potential of this junction with respect to the dipole midpotential is the average of the three limb electrode unipolar voltages, provided the three resistors are very large in comparison with the skin resistance and the internal resistance of the medium. Thus,

$$\begin{aligned} V_{CT} &= \frac{1}{3} (V_R + V_L + V_F) \\ &= \frac{1}{3} (\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3) \cdot \mathbf{p} \end{aligned} \quad (7)$$

The Wilson central-terminal voltage V_{CT} can be zero if the vector $\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3$ is zero or if $\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3$ is perpendicular to \mathbf{p} at every instant of time. The latter possibility must be discarded since the heart-vector variations are completely independent of the unipolar vectors \mathbf{c}_1 , \mathbf{c}_2 and \mathbf{c}_3 , and \mathbf{p} can be expected to be far from perpendicular to $\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3$ in the human subject.⁶ The former possibility is generally not realized in the human subject since the three unipolar vectors are independent; that is, when any factor upon which the unipolar vectors depend is changed, such as dipole position or inhomogeneities or shape of the medium, each unipolar vector changes in a way that is different from the changes in the others. Since there are definite indications^{6, 13, 21, 22} that the Wilson central-terminal voltage does not remain at the dipole midpotential during the cardiac cycle, it is not surprising that a more refined theory of electrocardiography would predict this.

DETERMINATION OF COEFFICIENTS

It can be seen from the mathematical relations presented that the general theory has been developed by introducing numerical coefficients which supply the connecting link between the heart dipole and the potentials it produces. In effect, the complexities of the electrical system have been lumped into and hidden in these constants. Since they play a

crucial role, a major effort must be made to determine them for individual subjects having variable body shapes, heart positions and inhomogeneities which vary in kind and distribution. At the present time model techniques are used primarily.^{1, 6, 23, 27}

Experimental Determination of Coefficients. Burger and van Milaan determined some of these coefficients by using a one-third scale model of the human torso (a statue) in which certain artificial inhomogeneities were introduced.¹ In part, they derived limb lead expressions for a dipole in the center of the heart. Their equations are, in terms of the rectangular coordinate system of figure 2,

$$\begin{aligned} V_I &= 78 p_x - 25 p_y + 20 p_z \\ V_{II} &= 30 p_x + 150 p_y - 18 p_z \end{aligned} \quad (8)$$

These results have been transposed into the same terms used in this laboratory. This involves a change of their coordinate system; a reversal of their unconventional definition of V_{II} , which is the negative of that given in equation 4; the use of our notation which avoids confusion between vectors and scalars; and the replacement of their multiplying factor of 10^{-5} by a factor 1.25. The last item permits direct comparison of the results and is allowable since only the relative values of the coefficient are of interest in this discussion.

Independent determinations of the coefficients have been made in this laboratory for the frontal plane using both male and female life-size homogeneous torso models.⁶ The unipolar coefficients were determined, whereas Burger and van Milaan used a bipolar technic which gave less information. The unipolar results for a dipole located in the center of the ventricular mass are

Male Torso

$$\begin{aligned} V_R &= -51p_x - 57p_y + 27p_z \\ V_L &= 25p_x - 84p_y + 41p_z \\ V_F &= -21p_x + 91p_y + 11p_z \end{aligned} \quad (9)$$

Female Torso

$$\begin{aligned} V_R &= -45p_x - 55p_y + 17p_z \\ V_L &= 32p_x - 73p_y + 25p_z \\ V_F &= -14p_x + 96p_y + 12p_z \end{aligned}$$

These unipolar results can be inserted into equation 4 to obtain the limb leads which can then be compared directly with equation 8. The result of this simple calculation is

$$\begin{aligned}
 & \textit{Male Torso} \\
 & V_I = 76p_x - 27p_y + 14p_z \\
 & V_{II} = 30p_x + 148p_y - 16p_z
 \end{aligned}
 \tag{10}$$

$$\begin{aligned}
 & \textit{Female Torso} \\
 & V_I = 77p_x - 18p_y + 8p_z \\
 & V_{II} = 31p_x + 151p_y - 5p_z
 \end{aligned}$$

It is hardly to be expected that these coefficients should agree with those in equation 8 since the models were different and the dipole positions, which exert a pronounced influence,^{1, 6, 13, 15} were not identical. However, by some remarkable coincidence, the agreement among the limb-lead torso coefficients is excellent. This is rather difficult to explain because Burger's torso was heterogeneous and ours were homogeneous. The greatest disagreement occurs in the p_z coefficients of the female torso which is only 9 per cent of the maximum coefficient of 150 units. The other limb-lead coefficients are in much closer agreement. The fact that the female torso coefficients show the largest deviations is not surprising since the shape of the female torso differed appreciably from that of the male.⁶

Future work will undoubtedly refine these results, but for the present there would seem to be far less error in using the above coefficients tentatively rather than the simplified theory based upon a spherical medium with a centric dipole.

As a matter of interest, the Wilson central-terminal voltage expressions for the male and female torsos, calculated from equation 7 using the unipolar voltages in equation 9, are given below.

$$\begin{aligned}
 & \textit{Male: } V_{CT} = -16p_x - 17p_y + 26p_z \\
 & \textit{Female: } V_{CT} = -9p_x - 10p_y + 18p_z
 \end{aligned}
 \tag{11}$$

The largest coefficient is 26 and this is about 17 per cent of the largest limb-lead coefficient. Thus, one would expect the Wilson central-terminal variations for this particular dipole position to be in the vicinity of 17 per cent of the maximum lead voltage,⁶ unless the di-

pole components display very abnormal behavior.

Theoretic Determination of Coefficients. The complexity of the human electrical system prohibits an exact determination of numerical coefficients by analytic methods. However, if the human body is represented by a simplified system, it can then be analyzed theoretically by known mathematical techniques. Its representation by a homogeneous conducting sphere with a centric dipole as shown in figure 3 is the classic theoretic model which has been used widely. In order to determine the coefficients for this theoretic model, one may start with the well known²¹ unipolar cosine functions:

$$\begin{aligned}
 V_R &= (3p/4\pi\gamma R^2) \cos(210^\circ - \alpha) \\
 V_L &= (3p/4\pi\gamma R^2) \cos(-30^\circ - \alpha) \\
 V_F &= (3p/4\pi\gamma R^2) \cos(90^\circ - \alpha)
 \end{aligned}
 \tag{12}$$

where p is the dipole moment, γ is the conductivity of the homogeneous sphere, R is the sphere radius and α is Einthoven's angle shown in figure 3. These unipolar voltages may be expressed in terms of the rectangular components of the dipole by trigonometric expansion. The result is

$$\begin{aligned}
 V_R &= -\sqrt{3} p_x - p_y \\
 V_L &= \sqrt{3} p_x - p_y \\
 V_F &= 2 p_y
 \end{aligned}
 \tag{13}$$

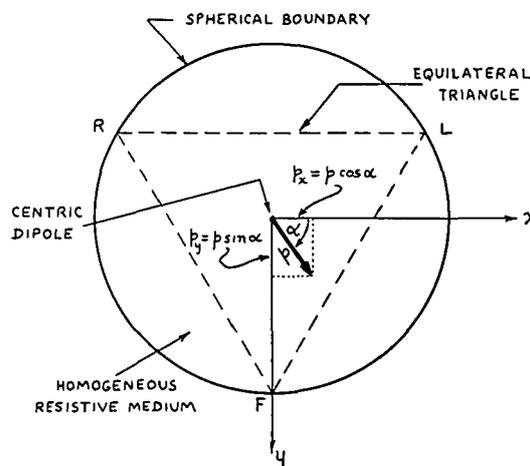


FIG. 3. Simplified theoretic model representing human torso as a homogeneous, resistive, spherical conducting medium at the center of which is an electric current dipole of moment p . Electrodes R, L, F are at corners of equilateral triangle whose plane is parallel to xy -plane of figure 2.

TABLE 1.—Comparison of Unipolar and Limb Lead Coefficients

Coefficient	c_{1z}	c_{1y}	c_{1z}	c_{2z}	c_{2y}	c_{2z}	c_{3z}	c_{3y}	c_{3z}	A_x	A_y	A_z	B_x	B_y	B_z
Torso															
Stature.....										78	-25	20	30	150	-18
Male.....	-51	-57	27	25	-84	41	-21	91	11	76	-27	14	30	148	-16
Female.....	-45	-55	17	32	-73	25	-14	96	12	77	-18	8	31	151	-5
Simple theory.....	-78	-45	0	78	-45	0	0	90	0	156	0	0	78	135	0

where $p_x = p \cos \alpha$, $p_y = p \sin \alpha$ (fig. 3) and the common factor $\frac{3}{4}\pi\gamma R^2$ has been assigned a numerical value of 2.0. This is allowable since only relative values are of interest here.

Several severe shortcomings of this simplified theory are obvious when equation 13 is compared with the experimental results that have been presented:

(a) The chest-to-back component of the dipole, p_z , which can have as much as 25 per cent influence on frontal-plane body surface voltages according to the torso data, plays no part in the simplified theory. Of course, the disappearance of p_z can be traced to the fact that the dipole has been assumed, unrealistically, to be at the center of a symmetric volume conductor.

(b) The voltages in equation 13 are not independent as can be seen by adding them; $V_R + V_L + V_F = 0$ for all values of p_x , p_y and, of course, p_z . Indeed, this result has been used to justify the use of the Wilson central terminal.²⁰ However, there is considerable evidence from a variety of sources, including the torso data presented here, that the Wilson central-terminal voltage is not zero, typical departures being 15 to 40 per cent of the maximum lead voltage.^{6, 13, 21, 22}

(c) The relative values of the coefficients are quite different from those which have been measured on torsos. These can be discussed in terms of the limb leads which, from equation 13, are

$$\begin{aligned} V_I &= V_L - V_R = 2\sqrt{3} p_x \\ V_{II} &= V_F - V_R = \sqrt{3} p_x + 3p_y \end{aligned} \quad (14)$$

It can be seen from equation 14 that according to the simple theory, V_I depends solely on p_x ; however, it is clear from equations 8 and 10 that p_y and p_z exert a sizeable influence on

V_I . For example, in equation 8 or 10, p_y contributes 30 to 35 per cent as much as p_x to V_I . The relative influence of p_y and p_x on V_{II} given by the simplified theory in equation 14 can be seen to have the ratio of $3:\sqrt{3}$; that is, the relative influence on V_{II} of p_y is about 1.7 times that of p_x . However, in the torso models the corresponding relative influence is on the order of 5:1, a pronounced difference from that expected from simple theory. Returning to V_L for a moment, a very marked departure is noticeable. The simple theory equation 13 shows that p_x influences V_L about 70 per cent more than does p_y ; the male model data in equation 9 indicate an opposite effect, that is, p_y influences V_L more than three times as much as p_x .

Table 1 shows a comparison of the unipolar and limb lead coefficients when the theoretic factor $\frac{3}{4}\pi\gamma R^2$ is arbitrarily assigned a value 45 so that relative numerical values may readily be seen.

It is clear that while the simplified theoretic model permits a mathematical analysis, the results are markedly different from those derived from the human torso. The errors of the simplified theory become even larger when compared with torso data involving a dipole that is not located in the center of the heart.^{1, 6}

By relaxing some of the restrictions on the simplified representation of figure 3, better agreement occurs between the theoretic model and the best available coefficients. For instance, the analysis of an eccentric dipole in a homogeneous conducting sphere has been presented.^{13, 25, 26} However, it is improbable that any simple theoretic model can give results that are as reliable as experimentally determined coefficients.

GEOMETRIC INTERPRETATION

It is possible to devise a geometric representation¹ of the general theory which is similar, in principle, to the commonly used geometric interpretation associated with centric dipole theory for a homogeneous conducting sphere. In fact, the general geometric representation includes, as a special case, the simplified vector-projection methods widely used today. It should be pointed out, however, that the geometric representation is not a necessary part of the theory; the entire matter can be expressed in algebraic equations of the form given in equation 1.

A geometric interpretation of equation 1, which has the general form of all equations in the general theory, is indicated by the vector form $V = \mathbf{c} \cdot \mathbf{p}$. The dot product of the two vectors \mathbf{c} and \mathbf{p} can be given the standard¹⁷ geometric interpretation; that is, the unipolar voltage V can be regarded as a scalar quantity arising from the projection of the time-varying heart vector \mathbf{p} onto the fixed vector \mathbf{c} , multiplied by the magnitude of \mathbf{c} . This interpretation is indicated diagrammatically in figure 4, which reveals the essence of the scheme. It should be noted that both \mathbf{p} and \mathbf{c} are vectors in three dimensions which are not confined to any of the three planes of figure 2.

An application of this geometric interpretation to the frontal-plane male torso data presented earlier is given in figure 5. The unipolar vectors, which generally do not form a triangle, are shown at the left drawn from an origin O representing the midpotential of the dipole. The tips of the vectors are labeled R' , L' and F' and are the points in "image" space²⁷ corresponding to the physical points R , L and F , respectively, on the human subject. The departure of these so-called image points from their physical counterparts on the human subject can be regarded as a measure of electrical distortion of dipole behavior as seen at the body surface. The limb-lead triangle whose sides are the limb-lead vectors \mathbf{A} , \mathbf{B} and \mathbf{C} is shown at the right; the origin of the coordinate system is arbitrary. While it follows from the definitions of \mathbf{A} , \mathbf{B} and \mathbf{C} that this triangle is formed by joining the tips of the

unipolar vectors it is shown separately in order to clarify the drawing. It should be noted that the limb-lead triangle is neither equilateral nor parallel to the frontal plane.

The procedure illustrated generally in figure 4 may be applied to the vectors in figure 5 in order to deduce the scalar voltages by geometric construction. For example, the method of obtaining lead II is indicated by equation 4, $V_{II} = \mathbf{B} \cdot \mathbf{p} = Bp \cos \theta_{Bp}$, where θ_{Bp} is the angle in space between \mathbf{B} and \mathbf{p} . Consequently, V_{II} can be obtained by projecting the heart-dipole vector \mathbf{p} on vector \mathbf{B} and multiplying $p \cos \theta_{Bp}$, the result of such projection, by the magnitude of \mathbf{B} . This procedure demonstrates that departure of the limb-lead triangle from the oversimplified equilateral case has two effects on the result; not only are the angles between \mathbf{p} and the triangle sides altered, but also the unequal lengths of the triangle sides introduce unequal weighting factors which in the equilateral simplification are all equal and hence ignored.

This type of geometric interpretation can be extended readily to the case of four or more

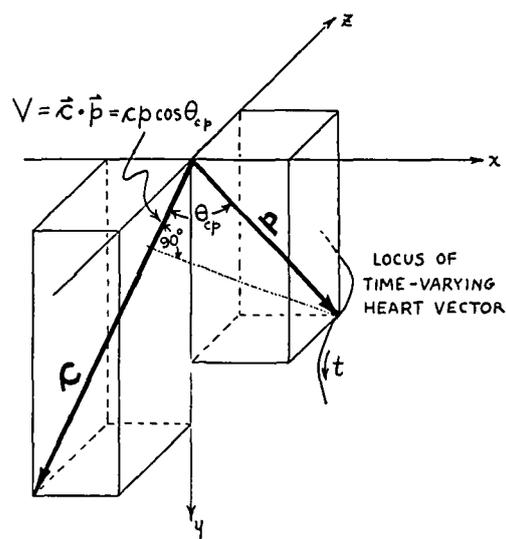


FIG. 4. Illustration showing essence of general vector-projection scheme forming basis of the geometric representation, in terms of a fixed unipolar vector \mathbf{c} and a heart-dipole vector \mathbf{p} of variable moment but fixed in position at the origin. Scalar unipolar voltage V can be obtained by projecting the heart dipole vector \mathbf{p} on vector \mathbf{c} , $p \cos \theta_{cp}$ being the result of this projection, and then multiplying by the magnitude of \mathbf{c} . (Discussion in text.)

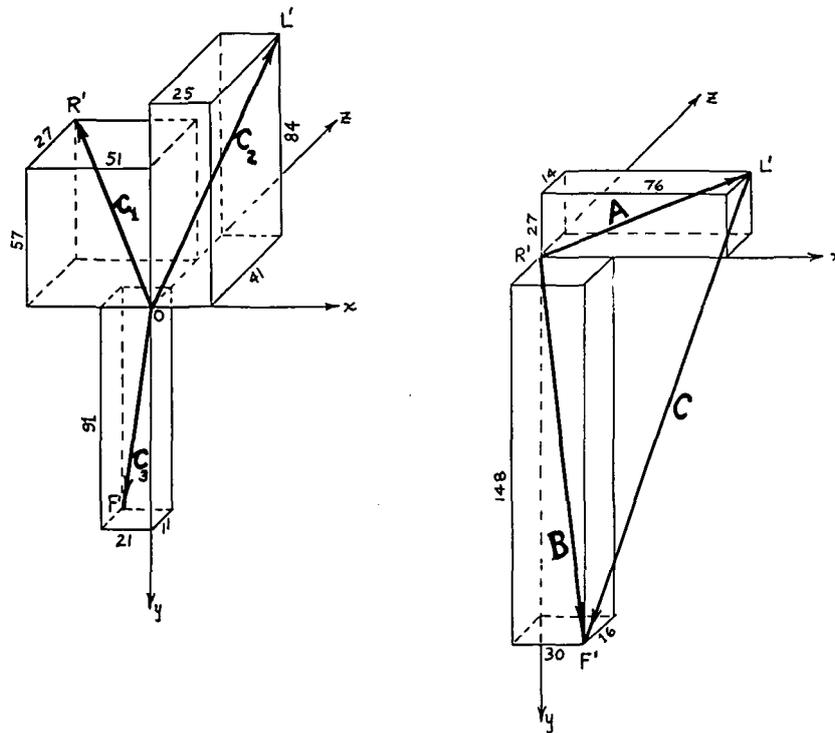


FIG. 5. Examples of geometric representation of experimental results, for a dipole in the anatomic center of the heart, to the frontal-plane male torso. At the left the unipolar vectors c_1 , c_2 and c_3 emanate from point O which represents the midpoint of the dipole. These unipolar vectors do not form a triangle since their sum is not zero. Tips of the vectors labeled R' , L' and F' are points in image space which correspond to the human-body points R , L and F , respectively. The limb lead triangle whose sides are the vectors A , B and C is formed by joining the tips of the vectors c_1 , c_2 and c_3 . This triangle is shown separately at the right with an arbitrary origin of the coordinate system. The magnitudes of the vectors A , B and C are 82, 152, and 184, respectively.

The three-dimensional aspect of these vectors is conveyed by use of rectangular solids, the numbers on which give relative numerical values of vector components [see equations 9 and 10]. Vector c_1 , directed upward toward the right and back; vector c_2 , directed upward toward the left and back; vector c_3 , directed downward toward the right and back. The plane of the limb-lead triangle is not parallel to the frontal plane; its lower apex, the tip of vectors B and C , lies in front of the xy -plane and its upper apex, the tip of vector A , lies to the rear of the xy -plane. However, the entire triangle lies behind the electrical mid-point of the heart dipole. Both diagrams drawn to the same relative voltage scale.

electrodes situated at any points on the human body. This follows naturally from the fact that equation 1 is applicable to any points within or on the body. For each electrode position there is a corresponding point in image space, as was the case for the points R' , L' and F' . However, the location of these points differ from their physical counterparts. Vectors connecting these points in image space have components equal to the coefficients in the mathematical expressions for the potential differences between the physical points, as was true in the frontal-plane example. These

vectors can be used to determine the scalar potential differences by projecting the heart-dipole vector p onto them and multiplying by the magnitude of the vector, as before.

An extension of this idea to all points on the surface of the human subject leads to the concept of a surface in image space each point on which corresponds to a physical point on the torso surface.¹ The shape of the image surface differs markedly from that of the human torso,²⁷ and the region outside the image surface corresponds to points within the human subject because the internal voltages are larger and

have larger coefficients with correspondingly larger fixed vectors.* In general, the projection of \mathbf{p} on any vector joining two points of the image surface times the length of the vector gives the potential difference measured on the human subject between body-surface points corresponding to the same image points which are joined by the vector.

Several important ideas emerge from this general geometric interpretation:

(a) *Choice of Body-Surface Electrodes.* Theoretically the locations of body-surface electrodes are immaterial as long as the corresponding points on the image surface are known, which is equivalent to knowing the coefficients entering into the equations for the electrode positions selected. In principle, any four different electrode positions are sufficient to determine uniquely the behavior of \mathbf{p} , provided all the pertinent coefficients are known. However, practical considerations such as relative magnitudes of the coefficients, reproducibility of electrode placement, and backlog of empiric data might strongly influence the choice of electrode positions.

(b) *Systems of Vectorcardiography.* Since the electrode positions are theoretically inconsequential from the standpoint of determining the heart vector, it might be expected that the results of various systems of vectorcardiography would all agree. However, they do not, and it is likely that this is so largely because inaccurate methods for analysis are being employed. In effect, it is assumed either that the image surface is a sphere or that it has the same shape as the human body, neither being the case.²⁷ For example, when four electrodes are arranged on the body, the heart-vector projection is performed by using lines joining the physical electrodes rather than

the vectors connecting points that correspond to these electrodes in image space. This is done for cube, tetrahedron and other electrode arrangements. This practice leads to errors on two counts, as mentioned earlier; first, the direction of the lines joining the physical electrodes is not correct for vector projection, and second, the weighting factors proportional to the different lengths of the constant vectors, even though the lengths joining the physical electrodes are equal, are not used. For instance, a physical cube arrangement on the human subject is distorted in image space just as the human triangle formed by the electrodes R , L and F was shown to be in figure 5. The conclusion seems obvious that the magnitude of errors introduced in various systems of vectorcardiography probably account for the differing interpretations with which we are presently confronted.

(c) *Specialized Electrode Positions.* The concept suggests that pairs of electrode positions can be located on the human subject so that the vectors joining the corresponding points in image space coincide with the principal axes x , y and z of figure 2. The use of electrodes so applied implies that all coefficients but one would be zero in each of the voltage equations for the potential difference between these electrodes. Therefore the measured voltages would be proportional to only one component of the heart vector. Each pair of electrodes would, however, have a different weighting factor unless the vectors in image space had the same length. The development of electrode positions of this type represents an interesting problem for future study, although some work has been done on it already.¹ The electrode positions would vary from one individual to the next, but the method of analyzing the data would be extremely simple in comparison with the three-dimensional problem posed by the use of "mixed" electrodes.

(d) *Improved Central Terminal.* The Wilson central-terminal voltage can be obtained by the same vector-projection method used for unipolar and limb-lead voltages. This is indicated by the vector dot-product form of equation 7, where the Wilson central-terminal vector is $\frac{1}{3}(\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3)$. It can be shown¹⁷

* The image-surface definition introduced by Burger and van Milaan has been adhered to here in order to avoid confusion. However, a more intuitively satisfying convention would be one which retains the present directions of the unipolar vectors but which defines their magnitudes inversely. This would make the physical inside of the human subject correspond to the inside of the image surface. Moreover, points of larger voltage, such as on the chest, would be located on a caved-in portion of the image surface rather than on an outward bulge as with the present definition.

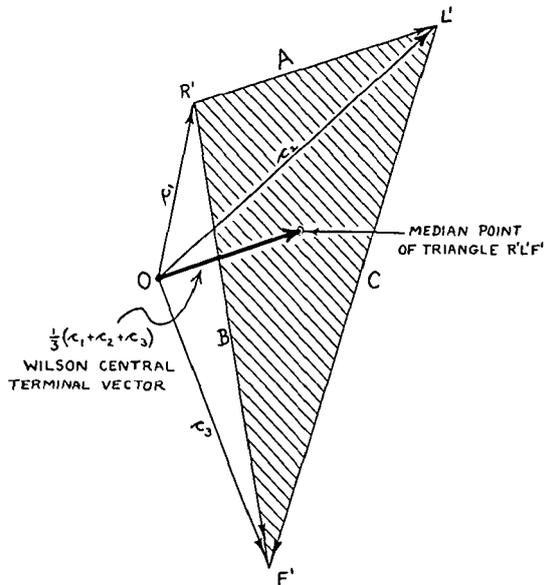


FIG. 6. Illustration of the general space-relationship between the Wilson central-terminal vector $\frac{1}{3}(c_1 + c_2 + c_3)$ and the unipolar and limb-lead vectors. The Wilson central-terminal vector is directed from the dipole midpotential point O to the median point of the shaded limb-lead triangle $R'L'F'$. This illustration attempts to convey the three-dimensional idea that O lies in front of the limb-lead triangle.

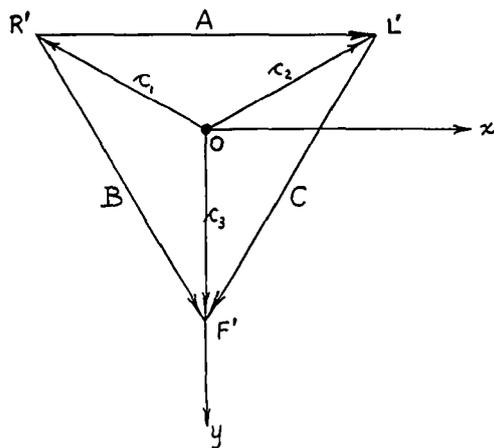


FIG. 7. Two-dimensional diagram of the unipolar and limb-lead vectors applicable to the simplified theoretical model of figure 3. (Discussion in text.)

that this Wilson central-terminal vector is directed from the origin O of figure 5 (left) to the median point of the limb-lead triangle connecting the tips of the unipolar vectors c_1 , c_2 and c_3 . This is illustrated in figure 6. The Wilson central-terminal vector, therefore indi-

cates by how much and in what direction the dipole midpotential deviates from the median point of the limb-lead triangle. However, the three-dimensional nature of the situation introduces some awkwardness, but this is also true of the other vectors as well.

Considering again the image surface associated with the actual surface of the human subject, it would seem feasible in principle to obtain a reference potential which is far more indifferent than the Wilson central terminal. This might be achieved by connecting properly adjusted resistors to body-surface points whose corresponding image points lie in a plane containing the dipole midpotential. Since the plane R', L', F' does not contain O , it follows that there is a non-zero lower limit to the value of the Wilson central-terminal voltage which is obtained when limb electrodes are employed, no matter what values of resistance are used.

When the general geometric interpretation is applied to the special case of a centric dipole in a homogeneous conducting sphere, the result commonly applied in frontal-plane electrocardiography is obtained. To show this, equations 13 and 14 may be written in vector dot-product form:

$$\begin{aligned}
 V_R &= (-\sqrt{3}i - j) \cdot p \\
 V_L &= (\sqrt{3}i - j) \cdot p \\
 V_F &= 2j \cdot p \\
 V_I &= 2\sqrt{3}i \cdot p \\
 V_{II} &= (\sqrt{3}i + 3j) \cdot p \\
 V_{III} &= (-\sqrt{3}i + 3j) \cdot p
 \end{aligned}
 \tag{15}$$

The familiar geometric representation of these equations is given in figure 7. The unipolar vectors $c_1 = -\sqrt{3}i - j$, $c_2 = \sqrt{3}i - j$ and $c_3 = 2j$ are each equal to 2.0 in magnitude, are coplanar, lie in the xy -plane (frontal plane) and in this special case form a triangle since $c_1 + c_2 + c_3 = 0$. The limb-lead vectors $A = 2\sqrt{3}i$, $B = \sqrt{3}i + 3j$ and $C = \sqrt{3}i + 3j$ are each equal to $2\sqrt{3}$ in magnitude, are in the same plane as the unipolar vectors, and form an equilateral triangle, the center of gravity of which coincides with the dipole midpotential point O . This last property shows geometrically that the Wilson central-terminal vector is zero. In brief, in this simplified case there is no distortion of the image surface compared with

the actual surface. This highly special result can be visualized as a degeneration of the frontal-plane geometric figure of figure 6, which is more nearly applicable to humans. Imagine that the point O moves to the center of gravity of the triangle R', L', F' whose sides at the same time become equilateral and whose plane shifts into the frontal plane in such a way that the vector A becomes parallel to the x-axis.

SUMMARY

A reiteration, formalization and expansion of the general mathematical and physical basis for heart-vector projection concepts is presented.

Specific equations are given, in terms of the general theory, for some commonly used electrodes in electrocardiography.

Experimental results obtained on human torso models are discussed in terms of heart-vector projection terminology.

The departures of presently used methods from torso experimental results is analyzed in quantitative mathematical and geometric terms.

A geometric interpretation of the general mathematical theory is presented and applied to unipolar leads, limb leads and the Wilson central-terminal voltage.

A geometric representation of the surface of the human body in image space is given and several ideas for possible future development are indicated.

The purpose of this study has been not only to encourage application of general theory, but also to emphasize the rather severe shortcomings of currently used methods of vector analysis.

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