Introduction to Quantum Information Science and Quantum Technologies

Assignment 1

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"It always seems impossible until it's done." - *Someone*

Question 1

Let us begin with the state,

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle .$$

which represents the superposition state of two basis states $\{|0\rangle, |1\rangle\}$ which are the computational or Zeeman (Z) basis. The coefficient c_0 and c_1 are the probability amplitudes for the states $|0\rangle$ and $|1\rangle$ respectively.

(a) Using the argument of orthogonality, where $\langle i|j\rangle = \delta_{ij}$, determine the inner-product $\langle 0|\psi\rangle$.

(b) Find the probability of measuring $|\psi\rangle$ in the states:

  (a) $|0\rangle$,

  (b) $\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$,

  (c) $\frac{1}{\sqrt{2}} |0\rangle + i\frac{1}{\sqrt{2}} |1\rangle$.

(c) If the quantum system we are working with is a photon, with the degree of freedom corresponding to the photon's path in the Mach-Zehnder interferometer, explain the answer to the last part.

Question 2

Extending on the previous part, let us define another state $|\psi\rangle$

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + c_1 |1\rangle ,$$

which represents the superposition state of two basis states $\{|0\rangle, |1\rangle\}$ where the probability amplitude coefficient is known.

(a) Using the argument of normalization, where $|c_0|^2 + |c_1|^2 = 1$, determine the probability amplitude coefficient for the state $|1\rangle$.

(b) Find the state $|\phi\rangle$, which is orthogonal to $|\psi\rangle$.

(c) Repeat the previous two parts for $|\psi_0\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ c_1 \end{pmatrix}$, using the vector notation. You can assume that the relative phase is zero.

(d) Another common set of basis is,

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Express the state $|\psi\rangle$ as a superposition of $|+\rangle$ and $|-\rangle$, which means find d_+ and d_- in,

$$|\psi\rangle = d_+ |+\rangle + d_- |-\rangle .$$

Question 3

Here I take a leap of faith and assume that you have successfully attempted both the previous questions. So let us now define a $|\psi\rangle$ which lives on the Bloch sphere,

$$|\psi\rangle = cos(\frac{\theta}{2}) |0\rangle + e^{i\phi} sin(\frac{\theta}{2}) |1\rangle .$$

(a) Explain, why the argument of cos and sin are $\theta/2$ instead of just θ.

(b) Construct the Bloch sphere and label the Z and X axis, along with the angles $\{\theta, \phi\}$ corresponding to each.

(c) Find the angles $\{\theta', \phi'\}$ for the state $|\psi_{orth}\rangle$ such that $|\psi_{orth}\rangle$ points exactly away from $|\psi\rangle$ on the Bloch sphere, which means that,

$$\langle\psi_{orth}|\psi\rangle = 0.$$

(d) Using the polar representation show mathematically that the states $|\psi\rangle$ and $|\psi_{orth}\rangle$ are orthogonal and are valid basis states. Meaning they can represent any quantum state that lives on the Bloch sphere.

Question 4

Let the matrix $\hat{U}$ be defined such that,

$$\hat{U} = cos(\theta)\hat{I} + isin(\theta)\hat{\sigma}_x.$$

(a) Prove that this is a unitary matrix.

(b) Prove that $\hat{U} = e^{i\theta\hat{\sigma}_x}$, and subsequently that $\hat{U}$ is unitary.

(c) Use the previous solution to show that $\hat{R}_y(\pi) \neq N$, where N is the quantum NOT gate.

Question 5

The Hadamard gate we used in the class represented the action of a 50:50 beamsplitter. From this, we get the relations,

$$\hat{U}_H \left|0\right\rangle = \frac{1}{\sqrt{2}} \left|0\right\rangle + \frac{1}{\sqrt{2}} \left|1\right\rangle,$$
$$\hat{U}_H \left|1\right\rangle = \frac{1}{\sqrt{2}} \left|0\right\rangle - \frac{1}{\sqrt{2}} \left|1\right\rangle.$$

.

According to these relations, the probability of measuring in $\hat{U}_H \left|0\right\rangle$ and $\hat{U}_H \left|1\right\rangle$ is 1/2. Let us try to reverse engineer this to construct the unitary $\hat{U}$ for a 30:70 beamsplitter.

(a) Construct any state $\left|\phi\right\rangle$ such that the probability of measuring $\left|0\right\rangle$ as the outcome in the Zeeman basis is $\frac{1}{3}$.

(b) Using this find the unitary matrix, such that $\hat{U}\left|0\right\rangle = \left|\phi\right\rangle$.

Question 6

(a) Construct the matrix form of $\hat{R}_{\frac{\vec{x}+\vec{z}}{\sqrt{2}}}(\pi)$ and show that this θ rotation qualifies as a legitimate single-qubit Hadamard gate.

(b) Construct the Hadamard as a combination of two rotations.

(c) Show that an $\hat{R}_y(\theta)$ rotation can be achieved by the sequence

$$\hat{R}_x(\pi)\hat{R}_y(-\theta)\hat{R}_x(-\pi),$$

where the order of implementation is right to left.

Question 7

A quantum computer programmer intends to transfer the initial state $\left|0\right\rangle$ to $\left|1\right\rangle$. She implements the transformation $\hat{R}_y(\pi)$ to achieve this. However, when she turns to her computer, she has errors, and the transformation that is really implemented is $\hat{R}_y(\theta + \varepsilon)$ where ε is the error and θ is her intended amount of rotation. The same applies to rotation about any axis and by any amount. So a nominal $\hat{R}_i(\theta)$ is really $\hat{R}_i(\theta + \varepsilon)$, where $i = x, y$.

(a) For a real erroneous 180° rotation, what is the probability that her final state is the desired $|1\rangle$ target state?

(b) She then decides to use three rotations instead of one. So she implements the rotations,

$$\hat{R}_y(\pi/2)\hat{R}_x(\pi)\hat{R}_y(\pi/2).$$

Does this sequence perform any better? Explain by drawing the trajectory on the Bloch sphere as well as mathematically. Make a plot of the performance metric as a function of ε.

Question 8

Let us implement our newly learned quantum mechanical skills on the Mach-Zehnder interferometer. In class, we defined the arms of the setup as the two basis states of our system, taking the beamsplitter to be a Hadamard gate and the mirror to have no impact on the system's state.

However, now let us take this same system but redefine the quantum states and see how the algebra and gates implemented in the setup change. Let us define the horizontally traveling photon as the ground state and the vertically traveling photon as the excited state.

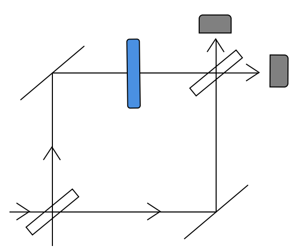


Figure 1: Schematic diagram for the Mach-Zehnder Interferometer.

(a) The beamsplitter under this redefinition is a Hadamard gate, but what gate is represented by the mirror? (Hint: It is called the NOT gate). Determine how its acts on the ground and excited state.

(b) Let the input for the setup be a single horizontal photon. What is the state after the first beamsplitter?

(c) What is the state after the mirrors?

(d) What is the state after the second beamsplitter? Which detector will detect the photon?

(e) Now, let us add a phase gate at the location shown in Figure 1 and re-evaluate the last part. Plot the probability of detection for both detectors against the phase induced by the phase gate.

Question 9

(a) Show that $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$ is indeed a solution to,

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

where $\hat{H}$ is time-independent.

(b) A qubit starts off as $|0\rangle$, implemented by a proton spin-1/2. Consider $\omega = \gamma B$ where $\gamma = 1$, ω is a (precessional) frequency, and B is a magnetic field. The desire is to do what is called a 'spin-echo' experiment. This experiment takes the state through the trajectory:

$$|0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \mapsto \frac{|0\rangle + e^{i\pi/3} |1\rangle}{\sqrt{2}} \mapsto \frac{|0\rangle + e^{-i\pi/3} |1\rangle}{\sqrt{2}} \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Draw a sequence of magnetic field pulses (say along x, y and z) that are required to achieve this spin-echo. Label the time-axis on the other pulse sequence.

Quantum Information Science
and Technology

Assignment 1: Solution.

Q1

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

a) $\langle 0|\psi\rangle = \langle 0|\left[c_0|0\rangle + c_1|1\rangle\right]$

$= c_0\langle 0|0\rangle + c_1\langle 0|1\rangle$

$= c_0(1) + c_1(0)$

$= c_0$

b)

i) $P(|0\rangle) = |\langle 0|\psi\rangle|^2$

$= |c_0|^2$

* Using part a)

ii) $P(|-\rangle) = |\langle -|\psi\rangle|^2$

where $|-\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$

$\langle -|\psi\rangle = \frac{1}{\sqrt{2}}\left[\langle 0| + \langle 1|\right]\left[c_0|0\rangle + c_1|1\rangle\right]$

$= \frac{1}{\sqrt{2}}\left[c_0 + c_1\right]$

$P(|-\rangle) = \frac{1}{2}|c_0 + c_1|^2$

iii) $P(|+i\rangle) = |\langle +i|\psi\rangle|^2$

where $|+i\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + i|1\rangle\right]$

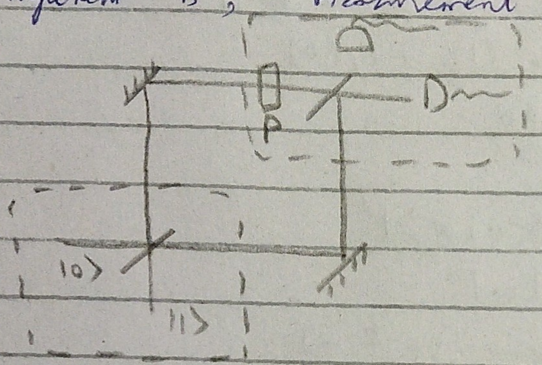
$\langle +i|\psi\rangle = \frac{1}{\sqrt{2}}\left[\langle 0| - i\langle 1|\right]\left[c_0|0\rangle + c_1|1\rangle\right]$

$= \frac{1}{\sqrt{2}}\left[c_0 - ic_1\right]$

$P(|+i\rangle) = \frac{1}{2}|c_0 - ic_1|^2$

c) The Mach-Zehnder interferometer, can be separated into different components,

Component A, Initialization
Component B, Measurement



The phase gate (retarder in component B) determines the basis of measurement.

Hence when the phase induced on state $|1\rangle$, (the upper arm) is 1, we get part i), if the phase is -1, we get part ii), for phase $+i$, we get part iii).

These measurements can also be translated to the polarization picture, where the basis states are $|H\rangle, |V\rangle = |0\rangle, |1\rangle$ and measuring in the digonal and circular basis will lead you to the results of part ii) and iii).

Q2 $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + c_1|1\rangle$

a) $\langle\psi|\psi\rangle = 1$

$$\left[\frac{1}{\sqrt{2}}\langle 0| + c_1^*\langle 1|\right]\left[\frac{1}{\sqrt{2}}|0\rangle + c_1|1\rangle\right] = 1$$

$$\frac{1}{2} + c_1^* c_1 = 1$$

$$|c_1|^2 = \frac{1}{2}$$

$$\Rightarrow \quad c_1 = \frac{e^{i\phi}}{\sqrt{2}}$$

b) let $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

and $\langle\phi|\phi\rangle = 1$: Normalized

$\langle\phi|\psi\rangle = 0$: Orthogonal to $|\psi\rangle$

$$\langle\phi|\psi\rangle = \frac{1}{\sqrt{2}}\left[\alpha^*\langle 0| + \beta^*\langle 1|\right]\left[|0\rangle + e^{i\phi}|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\alpha^* + \beta^* e^{i\phi}\right] = 0$$

$$\beta^* = -\alpha^* e^{-i\phi}$$

$$\Rightarrow |\phi\rangle = \alpha^*|0\rangle - \alpha^* e^{-i\phi}|1\rangle$$

$$= \alpha^*\left[|0\rangle - e^{-i\phi}|1\rangle\right]$$

Using Normalization

$$\langle\phi|\phi\rangle = |\alpha|^2\left[\langle 0| - e^{+i\phi}\langle 1|\right]\left[|0\rangle - e^{-i\phi}|1\rangle\right]$$

$$1 = |\alpha|^2[1+1]$$

$$\Rightarrow \alpha = \frac{e^{i\theta}}{\sqrt{2}}$$

Then

Global phase

$$|\phi\rangle = \frac{e^{i\phi}}{\sqrt{2}}\left[|0\rangle - e^{-i\phi}|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle - e^{-i\phi}|1\rangle\right]$$

c) Same as part b) only in vector notation

d) $$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{i\phi}|1\rangle\right]$$

$$|\psi\rangle = d_+|+\rangle + d_-|-\rangle$$

$$= \frac{d_+}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] + \frac{d_-}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[d_+ + d_-\right]|0\rangle + \frac{1}{\sqrt{2}}\left[d_+ - d_-\right]|1\rangle$$

- Comparing leads to

$$d_+ + d_- = 1 \quad , \quad d_+ - d_- = e^{i\phi}$$

$$d_+ = \frac{1+e^{i\phi}}{2} \quad , \quad d_- = \frac{1-e^{i\phi}}{2}$$

$$\Rightarrow \quad |\psi\rangle = \frac{1+e^{i\phi}}{2}|+\rangle + \frac{1-e^{i\phi}}{2}|-\rangle$$

$$= e^{i\phi/2}\left\{\left[\frac{e^{i\phi/2}+e^{-i\phi/2}}{2}\right]|+\rangle - \left[\frac{e^{i\phi/2}-e^{-i\phi/2}}{2}\right]|-\rangle\right\}$$

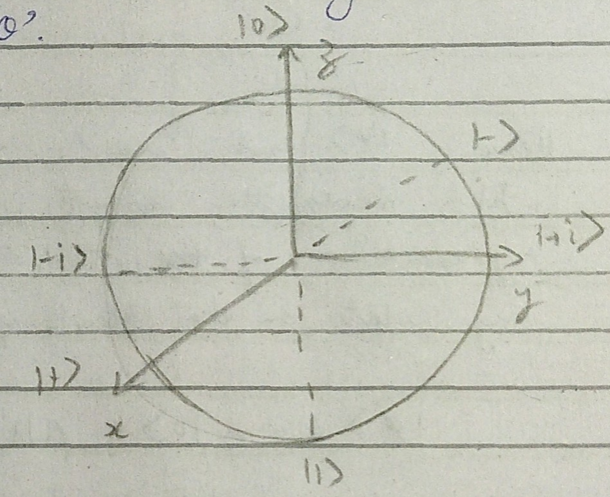
$$= \cos\left(\frac{\phi}{2}\right)|+\rangle + i\sin\left(\frac{\phi}{2}\right)|-\rangle$$

Q3

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

a) Using θ as the parameter of 'cos' and 'sin' function leads to degeneracy, the same state can be represented by 2 values of theta 'θ'.

b)



$|\psi(0,\phi)\rangle = |0\rangle$
$|\psi(\pi,\phi)\rangle = |1\rangle$

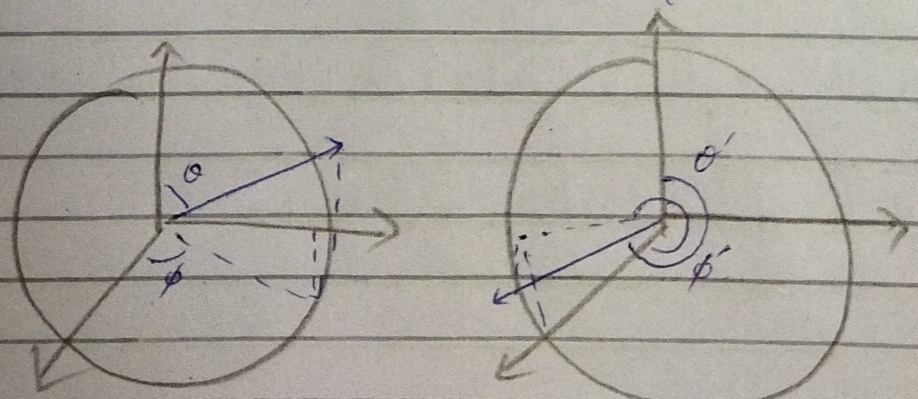
$|\psi(\pi/2, 0)\rangle = |+\rangle$
$|\psi(\pi/2,\pi)\rangle = |-\rangle$

c) To move a state $|\psi(\theta,\phi)\rangle$ to the opposite, we have to update the angles such that

$$|\psi(\theta,\phi)\rangle \longrightarrow |\psi_{orth}\rangle = |\psi(\pi-\theta, \pi+\phi)\rangle$$

$$\Rightarrow |\psi_{orth}\rangle = \cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)|0\rangle + e^{i\phi+i\pi}\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)|1\rangle$$

$$= \sin\left(\frac{\theta}{2}\right)|0\rangle - e^{i\phi}\cos\left(\frac{\theta}{2}\right)|1\rangle$$



d) $\langle \psi | \psi_{orth} \rangle = \left[\cos\left(\frac{\theta}{2}\right) \langle 0| + e^{-i\phi} \sin\left(\frac{\theta}{2}\right) \langle 1| \right] \cdot$

$$\left[\sin\left(\frac{\theta}{2}\right) |0\rangle - e^{i\phi} \cos\left(\frac{\theta}{2}\right) |1\rangle \right]$$

$$= \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)$$

$$= 0$$

Hence $|\psi\rangle$ and $|\psi_{orth}\rangle$ are orthogonal, which means they cannot written as a linear representation of the other and then represent any state on the bloch sphere.

$$|\phi\rangle = \alpha |0\rangle + \beta |1\rangle$$

$$|\phi\rangle = \alpha' |\psi\rangle + \beta' |\psi_{orth}\rangle$$

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Q4

$$\hat{U} = \cos(\theta)\hat{I} + i\sin(\theta)\hat{\sigma}_x$$

$$= \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$$

a) $$U^{\dagger} = \begin{bmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{bmatrix}$$

$$U^{-1} = \frac{1}{\cos^2\theta - i^2\sin^2\theta}\begin{bmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{bmatrix}$$

Since $U^{\dagger} = U^{-1}$, $\hat{U}$ is unitary

b) let $A = \exp\left[i\theta\hat{\sigma}_x\right]$

$$= 1 + i\theta\sigma_x + i^2\frac{(\theta)^2}{2!}\sigma_x^2 + \dots$$

Using $i^2 = -1$, $\hat{\sigma}_x^2 = \hat{I}$

$$= \hat{I} + i\theta\hat{\sigma}_x - \frac{\theta^2}{2!}\hat{I} - i\frac{\theta^3}{3!}\hat{\sigma}_x + \dots$$

$$= \hat{I}\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right] + i\hat{\sigma}_x\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right]$$

$$= \cos(\theta)\hat{I} + i\sin(\theta)\hat{\sigma}_x$$

$$= \hat{U}$$

Hence

$$\hat{U} = e^{i\theta\hat{\sigma}_x}$$

$$A = e^{i\theta\hat{\sigma}_x}$$

$$A^{+} = e^{-i\theta\hat{\sigma}_x} = \left[e^{i\theta\hat{\sigma}_x}\right]^{-1} = A^{-1}$$

Hence unitary.

c) Using

$$\hat{R}_{n\cdot\sigma}(\theta) = \cos\left(\frac{\theta}{2}\right)\hat{I} + i\sin\left(\frac{\theta}{2}\right) n\cdot\vec{\sigma}$$

$$\hat{R}_y(\pi) = \cos\left(\pi/2\right)\hat{I} + i\sin\left(\pi/2\right)\hat{\sigma}_y$$

$$= i\hat{\sigma}_y$$

$$= i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$U_{Not} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence $R_y(\pi) \neq U_{Not}$

* Note I have used $R_{n\cdot\sigma}(\theta) = e^{i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}}$

where it is actually $R_{n\cdot\sigma}(\theta) = e^{-i\theta/2\,\hat{n}\cdot\vec{\sigma}}$

Q5

a) $|\langle 0|\phi\rangle|^2 = \frac{3}{10}$

$\langle 0|\phi\rangle = \frac{e^{i\phi}}{\cancel{\sqrt{3}}}\sqrt{\frac{3}{10}}$

$\Rightarrow |\phi\rangle = \sqrt{\frac{3}{10}}\frac{e^{i\phi}}{\cancel{\sqrt{3}}}|0\rangle + c|1\rangle$

Using $\langle\phi|\phi\rangle = 1$

$|\phi\rangle = \cancel{\frac{1}{\sqrt{3}}|0\rangle}\sqrt{\frac{3}{10}}|0\rangle + e^{i\theta}\sqrt{\frac{7}{10}}|1\rangle$

b) $\hat{U}|0\rangle = \sqrt{\frac{3}{10}}|0\rangle + e^{i\theta}\sqrt{\frac{7}{10}}|1\rangle$

$\hat{U}|1\rangle = |\phi_{orth}\rangle$

Refer to Q2

$|\phi_{orth}\rangle = \sqrt{\frac{7}{10}}|0\rangle - e^{+i\theta}\sqrt{\frac{3}{10}}|1\rangle$

Then

$$U = \frac{1}{\sqrt{10}}\begin{bmatrix} \sqrt{3} & \sqrt{7} \\ e^{i\theta}\sqrt{7} & -e^{+i\theta}\sqrt{3} \end{bmatrix}$$

AnyScanner

Q.6

a) $$R_{\frac{\vec{x}+\vec{z}}{\sqrt{2}}}(\pi) = \cos\left(\frac{\pi}{2}\right)\hat{I} - i\sin\left(\frac{\pi}{2}\right)\left[\frac{\hat{\sigma}_x + \hat{\sigma}_z}{\sqrt{2}}\right]$$

$$= \frac{-i}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

global

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \hat{H}$$: Hadamard

b) let the rotations be in R_x and R_y

$$R_x(\theta)\, R_y(\phi) = \left[\cos\left(\frac{\theta}{2}\right)\hat{I} - i\sin\left(\frac{\theta}{2}\right)\hat{\sigma}_x\right] \cdot \left[\cos\left(\frac{\phi}{2}\right)\hat{I} - i\sin\left(\frac{\phi}{2}\right)\hat{\sigma}_y\right]$$

let $\cos\left(\frac{\theta}{2}\right) = c$, $\cos\left(\frac{\phi}{2}\right) = c'$

$$= \begin{bmatrix} c & -is \\ -is & c \end{bmatrix}\begin{bmatrix} c' & i^2 s' \\ -i^2 s' & c' \end{bmatrix}$$

$$= \begin{bmatrix} cc' - iss' & -cs' - ic's \\ -ic's + cs' & iss' + cc' \end{bmatrix}$$

Comparing elements to elements of Hadamard

$$cc' - iss' = 1/\sqrt{2} \quad (i) \qquad -cs' - ic's = 1/\sqrt{2} \quad (ii)$$
$$cs' - ic's = 1/\sqrt{2} \quad (iii) \qquad cc' + iss' = -1/\sqrt{2} \quad (iv)$$

Equation (i) + (iv)

$(iii) - (ii)$

$$\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi}{2}\right) = 0, \qquad \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi}{2}\right) = 0$$

Hence $\theta = \pi$

Then

$$R_x(\pi) R_y(\phi) = \overset{\nearrow \text{Global}}{-i} \begin{bmatrix} s' & c' \\ c' & -s' \end{bmatrix}$$

$$\sin\left(\frac{\phi}{2}\right) = \frac{1}{\sqrt{2}}, \qquad \cos\left(\frac{\phi}{2}\right) = \frac{1}{\sqrt{2}}$$

$\Rightarrow \phi = \pi/2$

Hence

$$\hat{U}_H = R_{\frac{\vec{x}+\vec{y}}{\sqrt{2}}} = R_x(\pi) R_y(\pi/2)$$

c) let

$$\begin{aligned} A &= R_x(\pi) R_y(\theta) R_x(-\pi) \\ &= [0 \cancel{-} i\sigma_x] R_y(\theta) [0 + i\sigma_x] \\ &= -i^2 \hat{\sigma}_x \left[\cos\left(\frac{\theta}{2}\right)\hat{I} - i\sin\left(\frac{\theta}{2}\right)\hat{\sigma}_y\right]\hat{\sigma}_x \\ &= \cos\left(\frac{\theta}{2}\right)\hat{\sigma}_x^2 - i\sin\left(\frac{\theta}{2}\right)\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x \\ &= \cos\left(\frac{\theta}{2}\right)\hat{I} + i\sin\left(\frac{\theta}{2}\right)\hat{\sigma}_y \\ &= R_y(-\theta) \end{aligned}$$

Q7

a) $P_{0\to1} = |\langle 1| R_y(\pi+\varepsilon)|0\rangle|^2$

where

$$R_y(\pi+\varepsilon) = \cos\left(\frac{\pi+\varepsilon}{2}\right)\hat{I} - i\sin\left(\frac{\pi+\varepsilon}{2}\right)\hat{\sigma}_y$$

$$= \left[\cos\left(\frac{\pi}{2}\right)\cdot\cos\left(\frac{\varepsilon}{2}\right) - \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\varepsilon}{2}\right)\right]\hat{I}$$

$$-i\left[\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\varepsilon}{2}\right) + \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\varepsilon}{2}\right)\right]\hat{\sigma}_y$$

$$= -\sin\left(\frac{\varepsilon}{2}\right)\hat{I} - i\cos\left(\frac{\varepsilon}{2}\right)\hat{\sigma}_y$$

$$R_y(\pi+\varepsilon)|0\rangle = -\sin\left(\frac{\varepsilon}{2}\right)|0\rangle - i^2\cos\left(\frac{\varepsilon}{2}\right)|1\rangle$$

$$\langle 1| R_y(\pi+\varepsilon)|0\rangle = +\cos\left(\frac{\varepsilon}{2}\right)$$

$$P_{0\to1} = \left|\cos\left(\frac{\varepsilon}{2}\right)\right|^2$$

$$= \frac{1}{2} + \frac{\cos(\varepsilon)}{2}$$

$$= 1 - \frac{\varepsilon^2}{4} + \dots$$

b) $P_{0\to1} = |\langle 1| R_y(\pi/2+\varepsilon)\, R_x(\pi)\, R_y(\pi/2+\varepsilon)|0\rangle|^2$

* Note: I have assumed R_x to be non-erroneous, the erroneous calculation is left for the students.

$$R_x(\pi) = -i\sigma_x = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$R_y\left(\frac{\pi}{2}+\varepsilon\right) = \left[\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\varepsilon}{2}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\varepsilon}{2}\right)\right]\hat{I}$$

$$-i\left[\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\varepsilon}{2}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\varepsilon}{2}\right)\right]\hat{\sigma}_y$$

let $\alpha = \cos\left(\frac{\varepsilon}{2}\right)$, $\beta = \sin\left(\frac{\varepsilon}{2}\right)$

$$R_y(\pi/2+\varepsilon) = \frac{(\alpha-\beta)}{\sqrt{2}} - i\frac{(\alpha+\beta)}{\sqrt{2}}\hat{\sigma}_y$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha-\beta & -(\alpha+\beta) \\ \alpha+\beta & \alpha-\beta \end{bmatrix}$$

$$R_y R_x = \frac{-i}{\sqrt{2}}\begin{bmatrix} -(\alpha+\beta) & \alpha-\beta \\ \alpha-\beta & \alpha+\beta \end{bmatrix}$$

$$R_y R_x . R_y = \frac{-i}{2}\begin{bmatrix} -(\alpha+\beta)(\alpha+\beta)+(\alpha-\beta)(\alpha+\beta) & (\alpha+\beta)^2+(\alpha-\beta)^2 \\ (\alpha-\beta)^2+(\alpha+\beta)^2 & -(\alpha+\beta)(\alpha-\beta)+(\alpha-\beta)(\alpha+\beta) \end{bmatrix}$$

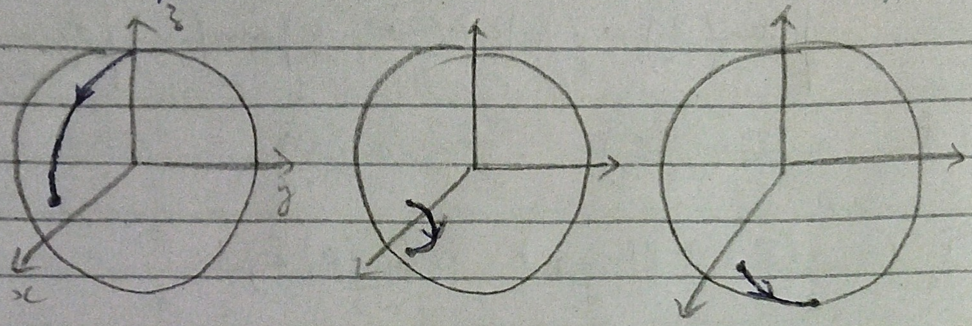
$$= \frac{-i}{2}\begin{bmatrix} 0 & 2(\alpha^2+\beta^2) \\ 2(\alpha^2+\beta^2) & 0 \end{bmatrix}$$

$$= -i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

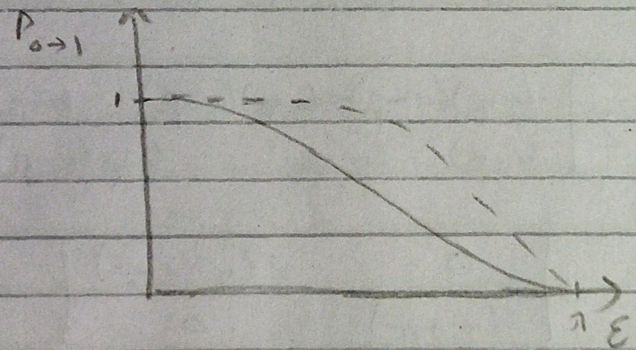
$$P_{0\to 1} = |\langle 1|R_y R_x R_y|0\rangle|^2$$
$$= |-i|^2$$
$$= 1$$

So, the error in the rotation have vanished, if $R_x(\pi)$ was erroneous $\Rightarrow R_x(\pi+\varepsilon)$, we would be left with an error smaller than what we had in part a).

- These rotations can be presented on the block sphere as



For part a) —
b) - - -



Q8

a) let the mirror be the gate U

$U|0\rangle = |1\rangle$: for lower arm
$U|1\rangle = |0\rangle$: for upper arm

$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, Hence it is a Not gate. $(\hat{X})$

b) $|\psi_{in}\rangle = |0\rangle$
$|\psi_1\rangle = \hat{H}|\psi_{in}\rangle$
$= \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$

c) $|\psi_2\rangle = \hat{X}|\psi_{in}\rangle$
$= \frac{1}{\sqrt{2}}[|1\rangle + |0\rangle]$

d) $|\psi_3\rangle = \hat{H}|\psi_2\rangle$
$= \frac{\hat{H}}{\sqrt{2}}[|0\rangle + |1\rangle]$

$= \frac{1}{2}[|0\rangle + |1\rangle + |0\rangle - |1\rangle]$

$= |0\rangle$

$P(D_0) = |\langle 0|\psi_3\rangle|^2$
$= 1$
$P(D_1) = |\langle 1|\psi_3\rangle|^2$
$= 0$

e) $|\psi_3\rangle = P|\psi_2\rangle$

$$= \left[e^{i\phi}|0\rangle\langle 0| + |1\rangle\langle 1|\right] \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{i\phi}|0\rangle + |1\rangle\right]$$

$|\psi_4\rangle = H|\psi_3\rangle$

$$= \frac{1}{2}\left[e^{i\phi}(|0\rangle+|1\rangle) + (|0\rangle - |1\rangle)\right]$$

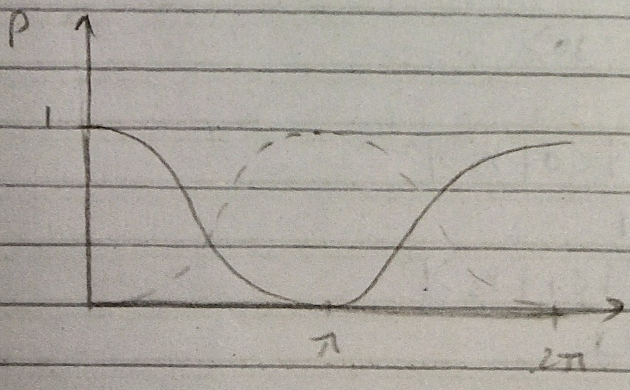
$$= \frac{1}{2}\left[(1+e^{i\phi})|0\rangle + (e^{i\phi}-1)|1\rangle\right]$$

$$= \frac{e^{i\phi/2}}{2}\left\{(e^{i\phi/2} - e^{-i\phi/2})|0\rangle + (e^{i\phi/2} - e^{-i\phi/2})|1\rangle\right\}$$

$$= e^{i\phi/2}\left[\cos\left(\frac{\phi}{2}\right)|0\rangle - i\sin\left(\frac{\phi}{2}\right)|1\rangle\right]$$

$P(D_0) = \cos^2\left(\frac{\phi}{2}\right)$: Rep ——

$P(D_1) = \sin^2\left(\frac{\phi}{2}\right)$: Rep - - -



Q9

a) $i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$

- Substituting $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$

$$i\hbar \frac{\partial}{\partial t} \left[e^{-i\hat{H}t/\hbar} |\psi(0)\rangle \right]$$

$$= i\hbar \left[\frac{\partial}{\partial t} e^{-i\hat{H}t/\hbar} \right] |\psi(0)\rangle$$

$$= i\hbar \left[\frac{-iH}{\hbar} \right] e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$$

$$= -i^2 \hat{H} e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$$

$$= \hat{H} \, e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$$

$$= \hat{H} |\psi(t)\rangle$$

Hence it is a solution.

b) For the first transition,

$$|0\rangle \xrightarrow{U_1} \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Rotation about y-axis by angle $\pi/2$

$$R_y(\pi/2) = e^{-i\frac{\pi}{4}\hat{\sigma}_y} = e^{-i\hat{H}_1 t/\hbar}$$

$$\frac{\hat{H}_1 t}{\hbar} = \frac{\pi}{4} \hat{\sigma}_y \frac{\omega}{\omega}$$

So $\hat{H}_1 = \hbar\omega\hat{\sigma}_y$, $t_{0\to 1} = \frac{\pi}{4\omega}$

For the second,

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{U_2} \frac{|0\rangle + e^{i\pi/3}|1\rangle}{\sqrt{2}}$$

Rotation about z-axis by an angle of $\pi/3$

$$R_z(\pi/3) = e^{-i\pi/6\,\hat{\sigma}_z} = e^{-i\hat{H}_2 t/\hbar}$$

$$\hat{H}_2 = \hbar\omega\hat{\sigma}_z \quad , \quad t_{1\to 2} = \frac{\pi}{6\omega}$$

For the third,

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{U_3} \frac{|0\rangle + e^{-i\pi/3}|1\rangle}{\sqrt{2}}$$

Rotation about x-axis by angle of π

$$R_x(\pi) = e^{-i\pi/2\,\hat{\sigma}_x} = e^{-i\hat{H}_3 t/\hbar}$$

$$\hat{H}_3 = \hbar\omega\hat{\sigma}_x \quad , \quad t_{2\to 3} = \frac{\pi}{2\omega}$$

For fourth,

$$\frac{|0\rangle + e^{-i\pi/3}|1\rangle}{\sqrt{3}} \xrightarrow{U_4} \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

- Using the similarity of transformation induced by U_2, we can recognise that
$$U_4 = U_2$$

$$\hat{H}_4 = \hbar\omega\sigma_z \quad , \quad t_{3\to 4} = \frac{\pi}{6\omega}$$

· Assumption

$|B_x| = |B_y| = |B_z| = \omega$