PHY 612 / CS 5112 / EE 539
Introduction to Quantum Information Science and Quantum Technologies

Mid-term Exam

15 March 2023 TIME ALLOWED: 1.5 HOURS

"Your mind is like water. When it is agitated it becomes difficult to see, but when you let it settle; the answer becomes clear." - *Oogway*

<u>INSTRUCTIONS TO CANDIDATES</u>

1. This examination paper contains five (5) questions.
2. Answer all questions. The marks for each question are indicated at the beginning of each question.
3. Add your ID and name to the answer booklets and label the booklets properly if you use more than one.
4. This exam is open class notes. You may not consult your assignments, book, or any other resources, online or otherwise.
5. Please write down systematically the steps in the workings.

Question 1. (10 marks)
In a hardware implementation, single qubit rotations about x and y are available, but one needs to construct a rotation through θ about the z axis. Propose and explain how an engineer can construct the operator,

$$\hat{U}(\theta) = exp(-i\theta\hat{\sigma}_z/2)$$

using the concatenation of the x and y rotation operators? Verify your answer.

Question 2. (10 marks)
Find the quantum Fourier transform of the three-qubit state,

$$|\phi\rangle = \frac{1}{2}(|1\rangle + |3\rangle + |5\rangle + |7\rangle)$$

where the j in $|j\rangle$ represents the integer associated with the 3 bit binary string?

Question 3. (10 marks)
Two types of GHZ states are given below:

$$\left|g^{+}\right\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$
$$\left|g^{-}\right\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle).$$

I intend to distinguish between the states $|g^{+}\rangle$ and $|g^{-}\rangle$ by merely measuring the **first qubit** in the $\{\ |0\rangle, |1\rangle\ \}$, i.e., Zeeman basis. What kind of a quantum circuit can be deployed to achieve this measurement?

Question 4. (10 marks)
A probabilistic teleportation circuit is shown in Figure 1. The circuit uses a $|W_3\rangle$ state, which is entangled and comprises 3 qubits. The unknown state is $|\phi\rangle$ and $|W_3\rangle$ takes the form:

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

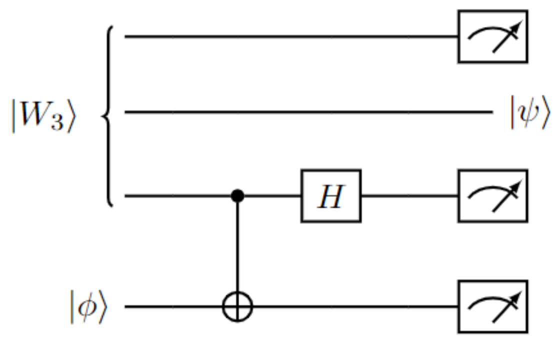

Figure 1: Quantum circuit for probabilistic teleportation.

The three detectors measure in the Zeeman basis, and the second qubit from the top bears the teleported state $|\psi\rangle$. Explain, using state progression, how the teleportation works and identify the success probability. What transformation would be needed to convert $|\psi\rangle$ to $|\phi\rangle$.

Question 5. (10 marks)

The Deutsch algorithm is implemented through the circuit of the form,

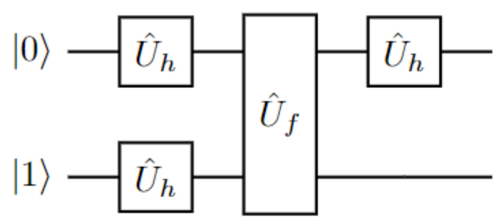

Figure 2: Quantum circuit for the Deutsch algorithm.

where $\hat{U}_H$ is the Hadamard gate and $\hat{U}_f$ is defined such that

$$\hat{U}_f |x\rangle |y\rangle = |x\rangle |f(x) \oplus y\rangle \, .$$

If the Hadamard gate is erroneous and performs the transformations:

$$\hat{U}_H |0\rangle = cos\theta |0\rangle + sin\theta |1\rangle ,$$
$$\hat{U}_H |1\rangle = sin\theta |0\rangle - cos\theta |1\rangle ,$$

where θ is slightly different from $\pi/4$, what is the probability that the oracle-based algorithm correctly determines f to be balanced?

END OF PAPER

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Q2.

$$\exp\left(-i\theta\,\frac{\hat{\sigma}_z}{2}\right) = \exp\left(+i\frac{\pi}{2}\frac{\hat{\sigma}_y}{2}\right)\exp\left(-i\theta\,\frac{\hat{\sigma}_x}{2}\right)\exp\left(-i\frac{\pi}{2}\frac{\hat{\sigma}_y}{2}\right)$$

Let's verify. Given

$$\exp(-i\beta A) = \cos\beta\;1 - i\sin\beta\;A \quad \text{when } A^2 = 1,$$

we have

$$\exp\left(-i\frac{\pi}{2}\frac{\hat{\sigma}_y}{2}\right).$$

As $\hat{\sigma}_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1,$

and likewise for $\hat{\sigma}_x$ and $\hat{\sigma}_y$, we obtain

$$\text{LHS} = \exp\left(-i\theta\,\frac{\hat{\sigma}_z}{2}\right) = \exp\left(-i\frac{\theta}{2}\hat{\sigma}_z\right)$$

$$= \cos\frac{\theta}{2}\;1 - i\sin\frac{\theta}{2}\;\hat{\sigma}_z$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}.$$

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1.2

Let's see the R.H.S.

$$\exp\left(-i\frac{\pi}{2}\frac{\sigma_y}{2}\right) = \exp\left(-i\frac{\pi}{4}\hat{\sigma}_y\right)$$

$$= \frac{1}{\sqrt{2}}\hat{1} - i\frac{1}{\sqrt{2}}\hat{\sigma}_y$$

$$\exp\left(-i\theta\frac{\hat{\sigma}_x}{2}\right) = \exp\left(-i\frac{\theta}{2}\hat{\sigma}_x\right)$$

$$= \cos\frac{\theta}{2}\,1 - i\sin\frac{\theta}{2}\hat{\sigma}_x$$

$i \cdot -i = +1$

$i \cdot i$

$i \cdot -i$

$-i \cdot -i$

$$\exp\left(+i\frac{\pi}{2}\frac{\sigma_y}{2}\right) = \exp\left(+i\frac{\pi}{4}\hat{\sigma}_y\right)$$

$$= \frac{1}{\sqrt{2}}\hat{1} + i\frac{1}{\sqrt{2}}\hat{\sigma}_y$$

Putting the terms together - the correct order:

$$RHS = \left(\frac{1}{\sqrt{2}}\hat{1} + i\frac{1}{\sqrt{2}}\hat{\sigma}_y\right)\left(\cos\frac{\theta}{2}\,1 - i\sin\frac{\theta}{2}\hat{\sigma}_x\right)\left(\frac{1}{\sqrt{2}}\,1 - i\frac{1}{\sqrt{2}}\hat{\sigma}_y\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} - i\sin\theta/2 & -\cos\frac{\theta}{2} - i\sin\theta/2 \\ -i\sin\theta/2 + \cos\frac{\theta}{2} & i\sin\theta/2 + \cos\theta/2 \end{pmatrix}$$

1.3 ص٣

$$= \frac{1}{2}\begin{pmatrix} 2\cos\frac{\theta}{2} - 2i\sin\frac{\theta}{2} & \cancel{-\cos\frac{\theta}{2} - i\sin\frac{\theta}{2} + i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \\ \cancel{-\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} - i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} & 2\cos\frac{\theta}{2} + 2i\sin\theta/2 \end{pmatrix}$$

$=$ LHS.

QED.

2.1

Q2

Multiple ways of attempting this. Here's one.

$$|\phi\rangle = \frac{1}{2}\left(|1\rangle + |3\rangle + |5\rangle + |7\rangle\right)$$

$$= \frac{1}{2}\left(|001\rangle + |011\rangle + |101\rangle + |111\rangle\right)$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{U}^{3\,qubits}_{QFT} = \frac{1}{\sqrt{8}} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ 1 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{pmatrix} \end{array}$$

where $\omega = e^{i\frac{2\pi}{8}} = e^{i\frac{\pi}{4}}$.

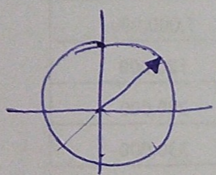
2.2

$\frac{2}{\sqrt{2^3}}$

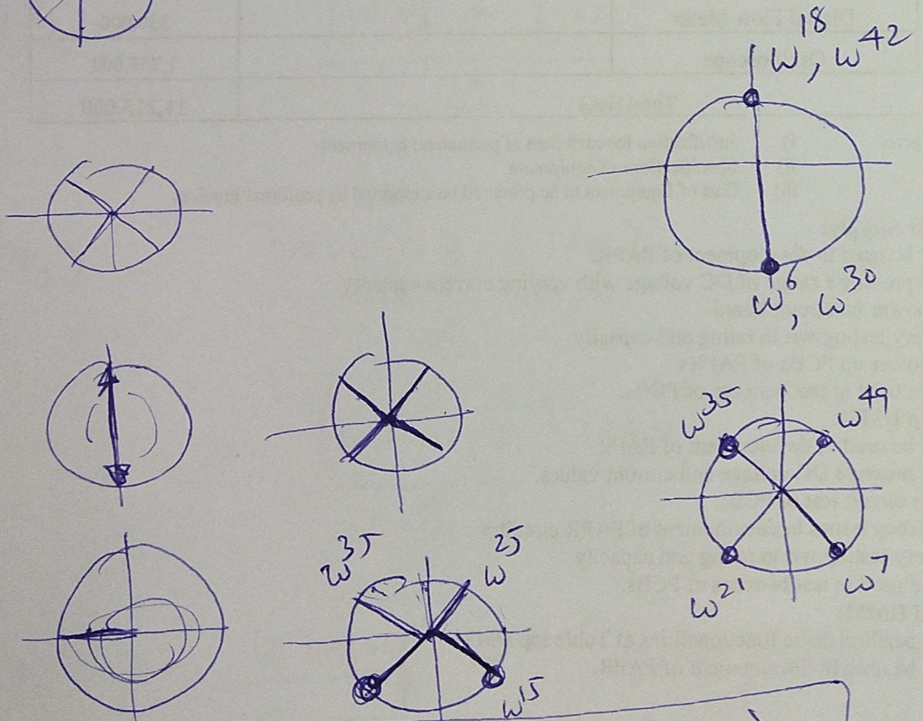
$$\hat{U}_a|\phi\rangle$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{8}}\begin{pmatrix} 1+1+1+1 \\ \omega+\omega^3+\omega^5+\omega^7 \\ \omega^2+\omega^6+\omega^{10}+\omega^{14} \\ \omega^3+\omega^9+\omega^{15}+\omega^{21} \\ \omega^4+\omega^{12}+\omega^{20}+\omega^{28} \\ \omega^5+\omega^{15}+\omega^{25}+\omega^{35} \\ \omega^6+\omega^{18}+\omega^{30}+\omega^{42} \\ \omega^7+\omega^{21}+\omega^{35}+\omega^{49} \end{pmatrix} \begin{matrix} \to 4 \\ \to 0 \\ \to 0 \\ \to 0 \\ \to -4 \\ \to 0 \\ \to 0 \\ \to 0 \end{matrix}$$

Now



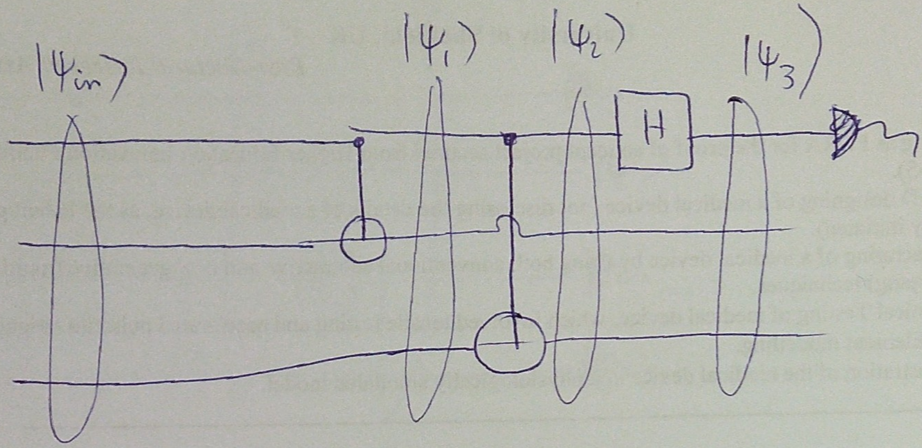
$\omega = e^{i\pi/4} = \angle 45^\circ$.



$$\boxed{\hat{U}_a|\phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |4\rangle\right)}$$

3.1

Q3



The above circuit is reminiscent of a Bell state measurement.

For $|\psi_{in}\rangle = |g^{\pm}\rangle = \dfrac{|000\rangle \pm |111\rangle}{\sqrt{2}}$

$$|\psi_1\rangle = \frac{|000\rangle \pm |101\rangle}{\sqrt{2}}$$

$$|\psi_2\rangle = \frac{|000\rangle \pm |100\rangle}{\sqrt{2}}$$

$$= \frac{(|0\rangle \pm |1\rangle)\,|00\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \underbrace{|0\rangle \otimes |00\rangle}_{\text{for } |g^+\rangle} \quad \text{or} \quad \underbrace{|1\rangle \otimes |00\rangle}_{\text{for } |g^-\rangle}$$

By measuring the 1st (top most qubit) in the Zeeman basis, one can distinguish between $|g^+\rangle$ and $|g^-\rangle$.

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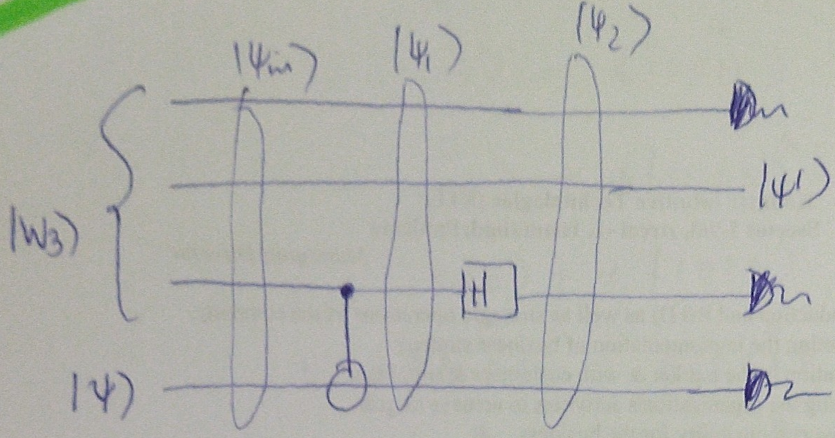
4.1



$$|W_3\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

$$|\psi_{in}\rangle = |W_3\rangle \otimes |\psi\rangle$$

$$= \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \otimes \left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= \frac{1}{\sqrt{3}}\Big[\alpha|0010\rangle + \alpha|0100\rangle + \alpha|1000\rangle + \beta|0011\rangle + \beta|0101\rangle + \beta|1001\rangle\Big]$$

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}\Big[\alpha|0011\rangle + \alpha|0100\rangle + \alpha|1000\rangle + \beta|0010\rangle + \beta|0101\rangle + \beta|1001\rangle\Big]$$

$$\begin{aligned}|\psi_2\rangle = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}\Big[& \alpha|00\rangle(|0\rangle - |1\rangle)|1\rangle \\ & + \alpha|01\rangle(|0\rangle + |1\rangle)|0\rangle \\ & + \alpha|10\rangle(|0\rangle + |1\rangle)|0\rangle \\ & + \beta|00\rangle(|0\rangle - |1\rangle)|0\rangle \\ & + \beta|01\rangle(|0\rangle + |1\rangle)|1\rangle \\ & + \beta|10\rangle(|0\rangle + |1\rangle)|1\rangle \Big]\end{aligned}$$

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4.2

$$|\psi_2\rangle = \frac{1}{\sqrt{6}}\Big[\alpha|0001\rangle - \alpha|0011\rangle + \alpha|0100\rangle + \alpha|0110\rangle + \alpha|1000\rangle + \alpha|1010\rangle + \beta|0000\rangle - \beta|0010\rangle + \beta|0101\rangle + \beta|0111\rangle + \beta|1001\rangle + \beta|1011\rangle\Big]$$

$$= \frac{1}{\sqrt{6}}\Big[|0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) \otimes |01\rangle \quad \leftarrow \text{combine 1st \& 9th terms}$$

$$+ |0\rangle \otimes (-\alpha|0\rangle + \beta|1\rangle) \otimes |11\rangle \quad \leftarrow \text{combine 2nd \& 10th term}$$

$$+ |0\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) \otimes |00\rangle \quad \leftarrow \text{combine 3rd \& 7th terms}$$

$$+ |0\rangle \otimes (\alpha|1\rangle - \beta|0\rangle) \otimes |10\rangle \quad \leftarrow \text{combine 4th \& 8th terms}$$

$$+ |1\rangle \otimes \Big\{\alpha|000\rangle + \alpha|010\rangle + \beta|001\rangle + \beta|011\rangle\Big\} \quad \leftarrow \text{all the rest.}$$

4.3 ص

The algorithm succeeds ~~with~~ when the 1st qubit is detected in the state $|0\rangle$. It is clear that this happens $8/12 = 2/3 \simeq 67\%$ of times.

The success prob is $2/3$.

By looking at the third and fourth qubits when the algo is successful, one can transform the 2nd qubit $|\psi'\rangle$ into $|\phi\rangle$ according to following table.

~~2n~~ 3rd qubit	4th qubit	$\hat{U}$ needed
0	0	$\hat{\sigma}_x$
0	1	$\hat{1}$
1	0	$\hat{\sigma}_z \hat{\sigma}_x = \hat{\sigma}_y$
1	1	$\hat{\sigma}_z$