Introduction to Quantum Information Science and Quantum Technologies

Assignment 2
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"Don't be afraid to fail, courage." - *Muriel*

Question 1

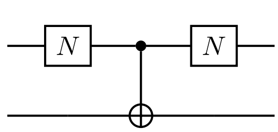
Given that $\hat{U}_H$ is the Hadamard gate, prove the following identities:

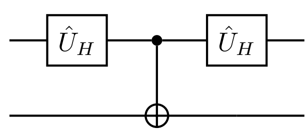
(a) $\hat{U}_H \hat{\sigma}_x \hat{U}_H = \hat{\sigma}_z$,

(b) $\hat{U}_H \hat{\sigma}_y \hat{U}_H = -\hat{\sigma}_y$,

(c) $\hat{U}_H \hat{\sigma}_z \hat{U}_H = \hat{\sigma}_x$.

Using these relations or otherwise, construct the truth table, matrix, and outer product form of the following circuits:





Question 2

Suppose in quantum computing hardware, only single qubit gates and a controlled $\hat{\sigma}_Z$ gate is possible (CZ), i.e.

$$CZ = |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_Z.$$

An engineer wants to implement a $CNOT$ gate using any single qubit gate and a CZ gate. How can she achieve that?

Question 3

The Toffoli gate is called the CCNOT gate since it has two qubits acting as controls and one target.

$$|a, b, c\rangle \xrightarrow{CCNOT} |a, b, ab \oplus c\rangle$$

(a) Make the matrix for CCNOT and write it in tensor form.

(b) Using this gate construct an Oracle $\hat{O}$ which induces the relative phase, $e^{i\pi} = -1$ to the $|00\rangle$, leaving the others unchanged.

(c) Using the rotation argument we used for the diffuser in Grover Algorithm, determine the state this Oracle $\hat{O}$ reflects about and corresponding tensor form.

(d) Construct another Oracle which induces the relative phase, $e^{i\pi} = -1$ to the states $|01\rangle$, and $|10\rangle$, leaving the other unchanged.

(e) Construct one last Oracle, which acts on three input qubits and induces the relative phase, $e^{i\pi/2} = i$ to the states $|000\rangle$, and $|111\rangle$, leaving the other unchanged.

Question 4

The no-cloning theorem is fundamental to quantum communication, which states that no unitary operation can allow one to clone a quantum system to another without disturbing the original system. Let us define,

$$|\phi\rangle = \alpha |0\rangle + \beta |1\rangle ,$$

where the probability amplitudes are unknown. and $|\phi\rangle$ cannot be recreated on a different channel.

(a) Suggest a method to measure α and list any approximations you make.

(b) Write the state $|\phi\rangle$ as a uniform superposition of the states $\{|+\rangle, |-\rangle\}$. How will your method of measuring α given this form evolve?

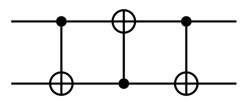
(c) Let us assume that we have a unitary $\hat{U}$, such that

$$\hat{U} |\phi\rangle \otimes |0\rangle = |\phi\rangle \otimes |\phi\rangle .$$

Prove the no-cloning theorem.

Question 5

Before we directly address the quantum teleportation circuit, we need to look at the SWAP gate, without which we would be unable to simulate our problems on the quantum computer.



(a) Write the matrix form of the SWAP gate.

(b) Determine the performance of this gate on the state $|\phi\rangle \otimes |\psi\rangle$. Where these states are defined as,

$$|\phi\rangle = \alpha |0\rangle + \beta |1\rangle ,$$
$$|\psi\rangle = \delta |0\rangle + \gamma |1\rangle .$$

(c) Distinguish this gate from the entanglement circuit.

Question 6

(a) Draw and describe a circuit that measures and distinguishes between the four Bell states:

$$|\phi\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),$$
$$|\psi\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

(b) Is the representation given below correct?

$$\hat{I} = |00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11| .$$

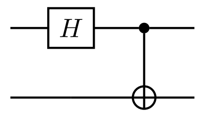
(c) Rewrite the $\hat{I}$ as a "mixture" of the four Bell states.

(d) Show that the Bell states are orthogonal w.r.t each other.

Question 7

Finally, we reach our first quantum algorithm, Quantum teleportation.

(a) Construct the truth table for the entanglement circuit.



(b) In class, we developed the quantum teleportation circuit for the initial state $|\phi+\rangle$. Repeat the derivation and design a new circuit for the state $|\psi+\rangle$.

Question 8

Using geometrical arguments, find the optimal number of Grover iterations needed to track a search state. Remember, the Grover operator is:

$$\hat{G} = (2\,|\psi\rangle\,\langle\psi| - \hat{I})\hat{O}_f.$$

Feel free to refer to the class notes.

QIT : Assignment 2
Solution Manual

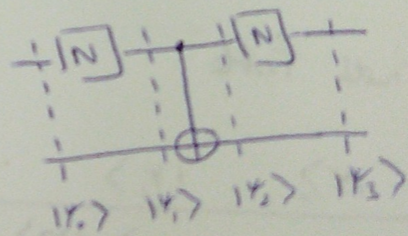
Q 1

a) $$U_H \hat{\sigma}_x U_H = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \hat{\sigma}_z$$

b) $$U_H \hat{\sigma}_y U_H = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{i}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{i}{2}\begin{bmatrix} 0 & 2 \\ -2 & \end{bmatrix} = -\hat{\sigma}_y$$

c) $$U_H \hat{\sigma}_z U_H = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \hat{\sigma}_x$$

d)



* Truth Table

ψ_0	ψ_1	ψ_2	ψ_3
00	10	11	01
01	11	10	00
10	00	00	10
11	01	01	11

$$U_{NCN} = |00\rangle\langle 01| + |01\rangle\langle 00| + |10\rangle\langle 10| + |11\rangle\langle 11|$$

- $U_{NCN} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $U_{NCN} = [\hat{N} \otimes \hat{I}][|0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{N}][\hat{N} \otimes \hat{I}]$

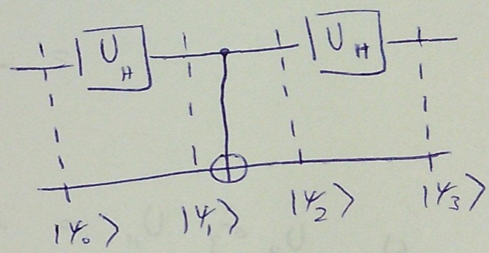
$= \hat{N}|0\rangle\langle 0|\hat{N} \otimes \hat{I} + \hat{N}|1\rangle\langle 1|\hat{N} \otimes \hat{N}$

$= |1\rangle\langle 1| \otimes \hat{I} + |0\rangle\langle 0| \otimes \hat{N}$

e) - $U_{HCH} = [\hat{U}_H \otimes \hat{I}] .$

$[|0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{N}] .$

$[\hat{U}_H \otimes \hat{I}]$



$= \hat{U}_H |0\rangle\langle 0| \hat{U}_H \otimes \hat{I} + \hat{U}_H |1\rangle\langle 1| \hat{U}_H \otimes \hat{N}$

$= |+\rangle\langle +| \otimes \hat{I} + |-\rangle\langle -| \otimes \hat{N}$

- Truth Table

$\|\psi\rangle_0$	$\|\psi_1\rangle$	$\|\psi_2\rangle$	$\|\psi_3\rangle$
$\|00\rangle$	$\frac{\|00\rangle + \|10\rangle}{\sqrt{2}}$	$\frac{\|00\rangle + \|11\rangle}{\sqrt{2}}$	$\frac{1}{2}[\|00\rangle + \|01\rangle + \|10\rangle - \|11\rangle]$
$\|01\rangle$	$\frac{\|01\rangle + \|11\rangle}{\sqrt{2}}$	$\frac{\|01\rangle + \|10\rangle}{\sqrt{2}}$	$\frac{1}{2}[\|00\rangle + \|01\rangle - \|10\rangle + \|11\rangle]$
$\|10\rangle$	$\frac{\|00\rangle - \|10\rangle}{\sqrt{2}}$	$\frac{\|00\rangle - \|11\rangle}{\sqrt{2}}$	$\frac{1}{2}[\|00\rangle - \|01\rangle + \|10\rangle + \|11\rangle]$
$\|11\rangle$	$\frac{\|01\rangle - \|11\rangle}{\sqrt{2}}$	$\frac{\|01\rangle - \|10\rangle}{\sqrt{2}}$	$\frac{1}{2}[-\|00\rangle + \|01\rangle + \|10\rangle - \|11\rangle]$

Then

$$U_{HCH} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

Q2

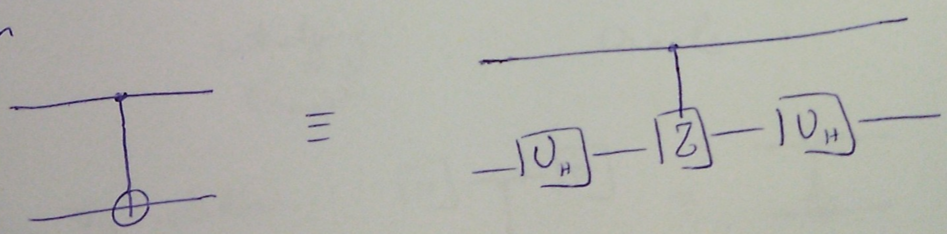
Using the relations

$$U_H U_H = U_H^{+} U_H = \hat{I}$$

$$U_H \hat{\sigma}_z U_H = \hat{\sigma}_x$$

$$\begin{aligned}
\text{CNOT} &= |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_x \\
&= |0\rangle\langle 0| \otimes U_H \hat{I} U_H + |1\rangle\langle 1| \otimes U_H \hat{\sigma}_z U_H \\
&= \hat{I}|0\rangle\langle 0|\hat{I} \otimes U_H \hat{I} U_H \\
&\qquad + \hat{I}|1\rangle\langle 1|\hat{I} \otimes U_H \hat{\sigma}_z U_H \\
&= [\hat{I} \otimes U_H][|0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_z][\hat{I} \otimes U_H] \\
&= [\hat{I} \otimes U_H]\, U_{cz}\, [\hat{I} \otimes U_H]
\end{aligned}$$

Then

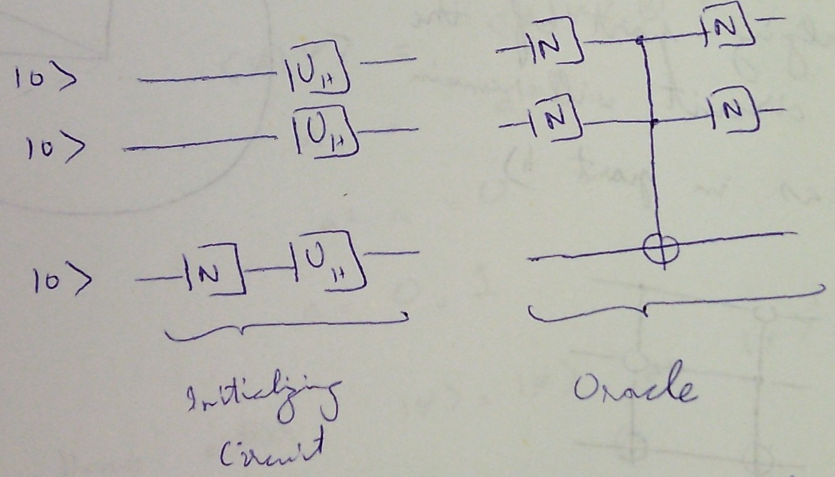


Q3

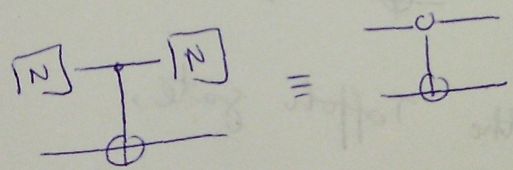
$$CCNOT = |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes \hat{\sigma}_x + \left[\hat{I} - |1\rangle\langle 1|\right] \otimes \left[\hat{I} - |1\rangle\langle 1|\right] \otimes \hat{I}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

b)



Using the notation from here onwards.



c) $\hat{O} = -|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|$

$$= -2|00\rangle\langle 00| + \underbrace{|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|}_{\hat{I}\otimes\hat{I}}$$

refer to Q6 b)

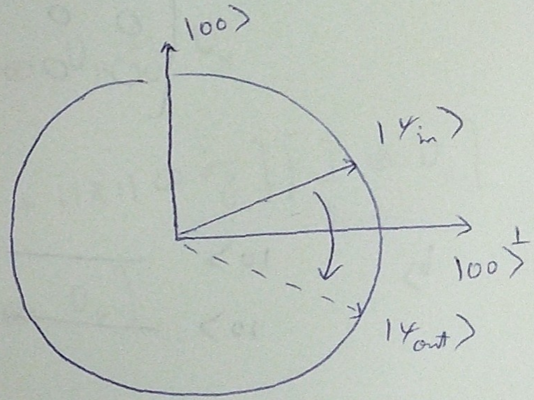
$$= -2|00\rangle\langle 00| + \hat{I}\otimes\hat{I}$$

$$= \hat{I}\otimes\hat{I} - 2|00\rangle\langle 00|$$

Then the rotation is about

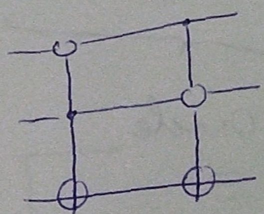
$$|00\rangle^{\perp} = \frac{1}{\sqrt{3}}\left[|01\rangle + |10\rangle + |11\rangle\right]$$

* Geometric Representation

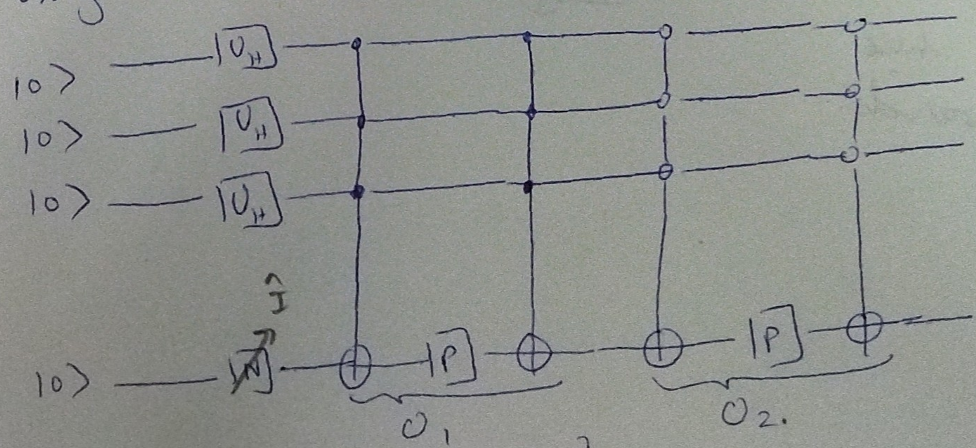


d) The initializing part of the quantum circuit will remain the same as in part b)

$\hat{O} =$



e) Using the Toffoli gate,



where $P = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{bmatrix}$

Q4

a) This part, I graciously leave to the students.

b) Refer to Q2 d) of Assignment 1.

c) let define

$$\langle\phi|\psi\rangle = a \quad , \quad U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$$

Then

$$\langle\phi|\langle 0| U^{\dagger} U|\psi\rangle|0\rangle = [U|\phi\rangle|0\rangle]^{\dagger}[U|\psi\rangle|0\rangle]$$

$$\langle\phi|\langle 0|\hat{I}|\psi\rangle|0\rangle = [|\phi\rangle|\phi\rangle]^{\dagger}[|\psi\rangle|\psi\rangle]$$

$$\langle\phi|\psi\rangle \otimes \langle 0|0\rangle = [\langle\phi|\langle\phi|][|\psi\rangle|\psi\rangle]$$

$$\langle\phi|\psi\rangle = \langle\phi|\psi\rangle \otimes \langle\phi|\psi\rangle$$

$$a = a^2$$

$$a^2 - a = 0$$

$$a = 0, 1$$

Hence either $|\psi\rangle = |\phi\rangle$ or $|\psi\rangle^{\perp} = |\phi\rangle$

Q5

a) $SWAP = \left[\left[|0\rangle\langle 0| \otimes \hat{I}\right] + |1\rangle\langle 1| \otimes \hat{N}\right].$

$\left[\hat{I} \otimes |0\rangle\langle 0| + \hat{N} \otimes |1\rangle\langle 1|\right] \cdot \left[|0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{N}\right]$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) $|\phi\rangle \otimes |\psi\rangle = \alpha\delta|00\rangle + \alpha\gamma|01\rangle + \beta\delta|10\rangle + \beta\gamma|11\rangle$

$$SWAP\,|\phi\rangle|\psi\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha\delta \\ \alpha\gamma \\ \beta\delta \\ \beta\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \alpha\delta \\ \beta\delta \\ \alpha\gamma \\ \beta\gamma \end{bmatrix}$$

$= \alpha\delta|00\rangle + \beta\delta|01\rangle + \alpha\gamma|10\rangle + \beta\gamma|11\rangle$

$= \delta|0\rangle \otimes [\alpha|0\rangle + \beta|1\rangle] + \gamma|1\rangle \otimes [\alpha|0\rangle + \beta|1\rangle]$

$= [\delta|0\rangle + \gamma|1\rangle] \otimes [\alpha|0\rangle + \beta|1\rangle]$

$= |\psi\rangle|\phi\rangle$

a) let $|\psi\rangle = |0\rangle$

such that

$$SWAP\ |\phi\rangle|0\rangle = |0\rangle|\phi\rangle$$

This is not dissimilar to the entanglement teleportation circuit which I can define as.

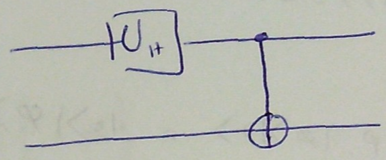
$$Tele\ |s\rangle|00\rangle = |ab\rangle|s\rangle$$

The only significant difference between the methods is that teleportation can be achieved irrespective of the distance between Alice and Bob but the SWAP gate assumes locality [due to the nature of controlled not gates.]

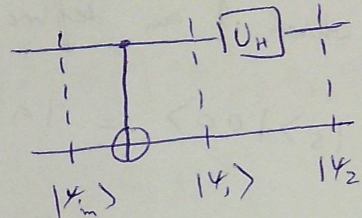
On a local computer such that Alice and Bob are using the same quantum hardware, the teleportation circuit is equivalent to the SWAP circuit.

Q6

a) Entangler circuit $\equiv$



Entangler$^+$ Circuit $\equiv$



- let the incoming states $|\psi_{in}\rangle$ be in the Bell basis.

Truth Table

$\|\psi_{in}\rangle$	$\|\psi_1\rangle$	$\|\psi_2\rangle$
$\|\phi_+\rangle$	$\frac{1}{\sqrt{2}}[\|00\rangle + \|10\rangle]$	$\|00\rangle$
$\|\phi_-\rangle$	$\frac{1}{\sqrt{2}}[\|00\rangle - \|10\rangle]$	$\|10\rangle$
$\|\psi_+\rangle$	$\frac{1}{\sqrt{2}}[\|01\rangle + \|11\rangle]$	$\|01\rangle$
$\|\psi_-\rangle$	$\frac{1}{\sqrt{2}}[\|01\rangle - \|11\rangle]$	$\|11\rangle$

Hence the Entangler$^+$ transformation allows us to measure the bell basis by their corresponding Zeeman basis.

b) $\hat{I} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|$

$= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|$

$= |0\rangle\langle 0| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|] + |1\rangle\langle 1| \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|]$

$= [|0\rangle\langle 0| + |1\rangle\langle 1|] \otimes [|0\rangle\langle 0| + |1\rangle\langle 1|]$

$= \hat{I} \otimes \hat{I}$, Hence True

c) Using the following

$$E_{nt} = [U_H \otimes \hat{I}][CN]$$

$$E_{nt}^{+} E_{nt} = E_{nt} E_{nt}^{+}$$

$$= [U_H \otimes \hat{I}][CN][CN][U_H \otimes \hat{I}]$$

$$= [U_H \otimes \hat{I}] \hat{I} [U_H \otimes \hat{I}]$$

$$= U_H U_H \otimes \hat{I}$$

$$= \hat{I} \otimes \hat{I}$$

$$= \hat{I}$$

And $E_{nt}|00\rangle = |\phi_+\rangle$

$E_{nt}|10\rangle = |\phi_-\rangle$

$\vdots$

Then

$$\hat{I} = E_{nt}^{+} E_{nt}$$

$$= E_{nt}^{+} \hat{I} E_{nt}$$

$$= E_{nt}^{+} [|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|] E_{nt}$$

$$= |\phi_+\rangle\langle\phi_+| + |\phi_-\rangle\langle\phi_-| + |\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-|$$

d) $$\langle \phi \pm | \psi \pm \rangle = \frac{1}{2}[\langle 00| \pm \langle 11|][|01\rangle \pm |10\rangle]$$

$$= \frac{1}{2}[\langle 00|01\rangle \pm \langle 00|10\rangle \pm \langle 11|01\rangle + \langle 11|10\rangle]$$

$$= 0$$

$$\langle \phi_+ | \phi_- \rangle = \frac{1}{2}[\langle 00| + \langle 11|][|00\rangle - |11\rangle]$$

$$= \frac{1}{2}[\langle 00|00\rangle - \langle 11|11\rangle] = 0$$

Similarly $\langle \psi_+ | \psi_- \rangle = 0$, Hence Bell states are orthogonal.

Q7

a) Refer to Q 6 a)

b) let

$$|s\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right]$$

$$|\psi_{in}\rangle = |s\rangle|01\rangle$$

$$|\psi_1\rangle = |s\rangle\, E_{nt_{12}}|01\rangle$$

$$= |s\rangle|\psi_+\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\alpha|0\rangle + \beta|1\rangle\right]\left[|01\rangle + |10\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\alpha|001\rangle + \alpha|010\rangle + \beta|101\rangle + \beta|110\rangle\right]$$

* Changing form befor applying the $E_{nt_{01}}^{+}$:

$$= \frac{1}{2\sqrt{2}}\Big\{\alpha|001\rangle + \alpha|111\rangle + \alpha|010\rangle + \alpha|100\rangle$$
$$+\alpha|001\rangle - \alpha|111\rangle + \alpha|010\rangle - \alpha|100\rangle$$
$$+\beta|101\rangle + \beta|011\rangle + \beta|110\rangle + \beta|000\rangle$$
$$+\beta|101\rangle - \beta|011\rangle + \beta|110\rangle - \beta|000\rangle\Big\}$$

$$= \frac{1}{2}\Big\{\frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] \otimes \left[\alpha|1\rangle + \beta|0\rangle\right]$$
$$+ \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right] \otimes \left[\alpha|1\rangle - \beta|0\rangle\right]$$
$$+ \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right] \otimes \left[\alpha|0\rangle + \beta|1\rangle\right]$$
$$+ \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right] \otimes \left[\alpha|0\rangle - \beta|1\rangle\right]\Big\}$$

$$\Rightarrow |\psi_1\rangle = \frac{1}{2}\Big[|\phi_+\rangle \otimes \hat{N}|s\rangle + |\phi_-\rangle \otimes \hat{N}\hat{Z}|s\rangle + |\psi_+\rangle \otimes |s\rangle + |\psi_-\rangle \otimes \hat{Z}|s\rangle\Big]$$

$$|\psi_2\rangle = Ent^{\dagger}_{01}\,|\psi_1\rangle$$

$$= \frac{1}{2}\, Ent^{\dagger}_{01}\Big[|\phi_+\rangle \otimes \hat{N}|s\rangle + |\phi_-\rangle \otimes \hat{N}\hat{Z}|s\rangle \; |\psi_+\rangle \otimes |s\rangle + |\psi_-\rangle \otimes \hat{Z}|s\rangle\Big]$$

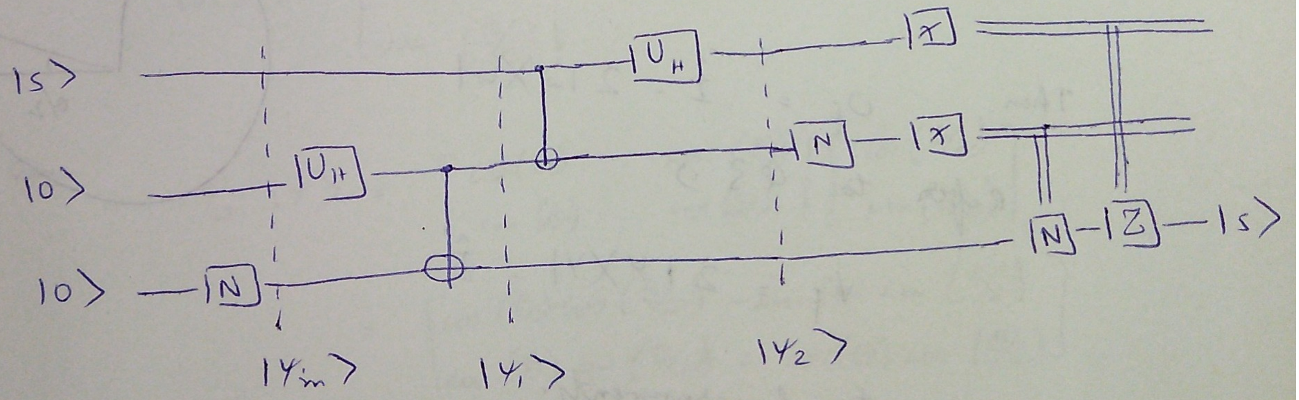
$$= \frac{1}{2}\Big[|00\rangle \otimes \hat{N}|s\rangle + |10\rangle \otimes \hat{N}\hat{Z}|s\rangle + |01\rangle \otimes |s\rangle + |11\rangle \otimes \hat{Z}|s\rangle\Big]$$

- Comparing states with operators

If Qubit 0 is in $|1\rangle$ apply $\hat{Z}$ on Qubit 2

If Qubit 1 is in $|0\rangle$ apply $\hat{N}$ on Qubit 2

- Hence we can make the following circuit.



Q 8

let $|\psi\rangle = H^{\otimes n} |0\rangle^{\otimes n}$

$$= H|0\rangle \otimes H|0\rangle \otimes \dots\dots H \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{N-1} |x\rangle$$

let the marked state be $|w\rangle$

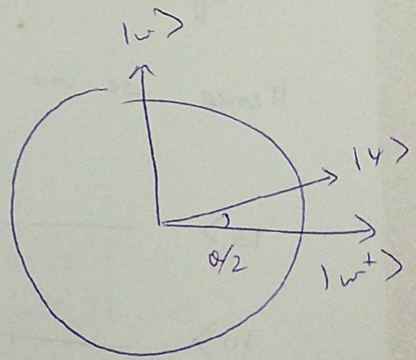
$$= \frac{1}{\sqrt{N}} \sum_{x \neq w} |x\rangle + \frac{1}{\sqrt{N}} |w\rangle$$

$$= \frac{1}{\sqrt{N}} \sqrt{\frac{N-1}{N-1}} \sum_{x \neq w} |x\rangle + \frac{1}{\sqrt{N}} |w\rangle$$

$$= \sqrt{\frac{N-1}{N}} |w^\perp\rangle + \frac{1}{\sqrt{N}} |w\rangle$$

Such that $\langle w^\perp | w\rangle = 0$

$\langle w^\perp | w^\perp\rangle = 1$



Then $O_f = \hat{I} - 2|w\rangle\langle w|$

Refer to Q 3 c)

$$V_f = 2|\psi\rangle\langle\psi| - \hat{I}$$

Using geometrical arguments.

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|w^\perp\rangle + \sin\left(\frac{\theta}{2}\right)|w\rangle$$

where $c = \cos(\theta/2) = \sqrt{\frac{N-1}{N}}$

$s = \sin(\theta/2) = \frac{1}{\sqrt{N}}$

Then $$V_f = 2\left[c|w^\perp\rangle + s|w\rangle\right]\left[c\langle w^\perp| + s\langle w|\right] - \hat{I}$$

where $$\hat{I} = |w^\perp\rangle\langle w^\perp| + |w\rangle\langle w|$$

$$= 2\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2c^2-1 & 2cs \\ 2cs & 2s^2-1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

$$O_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

So $$\hat{G} = V_f O_f = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$$

Then $$\hat{G}|\psi\rangle = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\cos(\theta/2) - \sin(\theta)\sin(\theta/2) \\ \sin(\theta)\cos(\theta/2) + \cos(\theta)\sin(\theta/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + \theta/2) \\ \sin(\theta + \theta/2) \end{bmatrix}$$

So

$$\hat{G}^r|\psi\rangle = \begin{bmatrix} \cos(r\theta + \theta/2) \\ \sin(r\theta + \theta/2) \end{bmatrix}$$

The solution state is after r iteration of $\hat{G}$

$$|\phi\rangle = |w\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\sin\left(r\theta + \theta/2\right) = 1$$

$$r\theta + \theta/2 = \pi/2$$

$$\theta\left[r + 1/2\right] = \pi/2$$

$$r + 1/2 = \frac{\pi}{2\theta}$$

$$r = \frac{\pi}{2\theta} - \frac{1}{2}$$

Using $\sin\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{N}}$

$$\theta = \lim_{N \to \infty} 2 \sin^{-1}\left(\frac{1}{\sqrt{N}}\right)$$

$$\Rightarrow \theta = \frac{2}{\sqrt{N}}$$

$$r = \frac{\pi}{2\left(2/\sqrt{N}\right)} - 1/2$$

$$r = \frac{\pi\sqrt{N}}{4} - 1/2$$

- Hence we have the exact number of iterations, but since the Oracle $\hat{G}$ can only implemented in integer form, we have to approximate r to the nearest integer which implies that

$$\| \langle w | G^{r_i} | \psi \rangle \|^2 < 1$$

- Try to find iterations for M solutions.