# Introduction to Quantum Information Science and Quantum Technologies

Assignment 4

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"What a fine day for science." - *Dexter*

## Question 1

(a) Prove using the L'Hopital's rule that

$$\lim_{x \to 0} xln(x).$$

(b) Prove from the first principle definition that the conditional entropy of event $B$ given $A$ is

$$H(B|A) = H(A, B) - H(A).$$

## Question 2

The Shannon entropy is given by

$$H(A) = -\sum_i p(a_i) logp(a_i).$$

(a) Show, using the Lagrange multiplier method, that entropy is maximized when $p(a_i) = 1/d$.

(b) If the probabilities are replaced by

$$p(a_i^{'}) = \sum_j \lambda_{ij} p(a_i), \qquad \lambda_{ij} \in \{0, 1\}$$

use the concavity of the function $f(x) = -xlog(x)$ to show that

$$H(A') \geq H(A).$$

## Question 3

(a) Show that $\hat{\rho}$ is a positive operator which means that $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for any $|\psi\rangle$.

(b) Show that its eigenvalues $\rho_m$ are real and positive. Hence

$$\rho = \sum_m \rho_m \,|\rho_m\rangle\,\langle\rho_m|$$

where $|\rho_m\rangle$ are the corresponding eigenvectors.

(c) How does the purity of a quantum state change as it evolves under a unitary operation?

## Question 4

Any density matrix can be written as,

$$\hat{\rho} = \frac{1}{2}(\hat{I} + \overrightarrow{r} \cdot \overrightarrow{\sigma})$$

where $\overrightarrow{r} = \{r_x, r_y, r_z\}$ and $\overrightarrow{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$.

(a) Find $\overrightarrow{r}$ given that $\hat{\rho} = |+\rangle\,\langle+|$ using the trace operator. Where the state $|+\rangle$ is defined as,

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

(b) Derive the expectation values of $\hat{\rho}$ for the operators $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

(c) Show that if $\hat{\rho}$ is a pure state, the trace of $\hat{\rho}^2$ is always unity.

## Question 5

For what value of $\varepsilon$ will the state

$$\hat{\rho} = (1-\varepsilon)\,|00\rangle\,\langle 00| + \varepsilon\,\left|\phi^+\right\rangle\left\langle\phi^+\right|$$

be entangled? Use the positivity of the partial transpose test. Here $|\phi^+\rangle$ is bell state.

## Question 6

What is the purity of the Weiner state defined as

$$\hat{\rho} = (1-\varepsilon)\frac{\hat{I}}{4} + \varepsilon\,\left|\psi^-\right\rangle\left\langle\psi^-\right|$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is the singlet state?

## Question 7

(a) Show that for the density matrix formalism, the density matrix evolves under a time-independent Hamiltonian as per the Liouville–von Neumann equation,

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}].$$

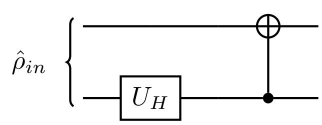
(b) Hence solve for $\hat{\rho}(t)$ given,

$$\hat{\rho}(0) = \frac{\hat{I} + \sigma_x}{2},$$
$$\hat{H} = \frac{\hbar w \sigma_z}{2}.$$

(c) Plot $\overrightarrow{r}$ against time for $\hat{\rho}(t)$ and represent this evolution on the Bloch sphere.

## Question 8

Consider the following circuit with $\hat{\rho}_{in} = 3/4\,|00\rangle\,\langle 00| + 1/4\,|11\rangle\,\langle 11|$.




Find the density matrix after the Bell creation circuit. What is the reduced matrix for the second qubit? (Trace out the first qubit).

Q1

a) $\lim_{x \to 0} x \ln x$

$$= \lim_{x \to 0} \frac{\ln(x)}{1/x}$$

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)}$$

$$= \lim_{x \to 0} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0} -x$$

$$= 0$$

b) $H(B|A) = \sum_i P(a_i)\ H(B|A=a_i)$

$$= -\sum_i P(a_i) \sum_j P(b_j|a_i) \log P(b_j|a_i)$$

$$= -\sum_{ij} P(b_j|a_i) P(a_i) \log P(b_j|a_i)$$

$$= -\sum_{ij} P(a_i, b_j) \log P(b_j|a_i)$$

$$= -\sum_{ij} P(a_i, b_j) \log \left[\frac{P(a_i, b_j)}{P(a_i)}\right]$$

$$= -\sum_{ij} P(a_i, b_j) \log P(a_i, b_j) + \sum_i \left(\sum_j P(a_i, b_j)\right) \log P(a_i)$$

$$= -\sum_{ij} P(a_i, b_j) \log P(a_i, b_j) + \sum_i P(a_i) \log P(a_i)$$

$$= H(A, B) - H(A)$$

Q 2

a) A has d possible outcomes

$$H(A) = -\sum_{i=1}^{d} P(a_i) \log P(a_i)$$

where $\sum_{i=1}^{d} P(a_i) = 1$ is a constraint

$$\frac{d}{dP(a_i)} \left[ H(A) + \lambda \left( \sum_i P(a_i) - 1 \right) \right] = 0$$

$$\frac{d}{d(P(a_i))} \left[ -\sum_i P(a_i) \log P(a_i) + \lambda \left[ \sum_i P(a_i) - 1 \right] \right] = 0$$

$$-\sum_i \left[ -\frac{P(a_i)}{P(a_i)} + \log P(a_i) + \lambda \right] \frac{d}{dp} P(a_i) = 0$$

$$\Rightarrow \quad -1 + \lambda + \log P(a_i) = 0$$

$$\log P(a_i) = 1 - \lambda$$

$$P(a_i) = 2^{1-\lambda}$$

$$\sum_i P(a_i) = \sum_{i=1}^{d} 2^{1-\lambda} = d(2^{1-\lambda}) = 1$$

$$2^{1-\lambda} = 1/d$$

$$1 - \lambda = \log\left(\frac{1}{d}\right)$$

$$1 - \lambda = -\log(d)$$

$$1 + \log(d) = \lambda$$

Hence $P(a_i) = 2^{1-\lambda} = 2^{-\log d} = 2^{\log(1/d)}$

$$= 1/d$$

So $H(A)$ is maximized for a uniform distribution.

b) ~~Thus~~ From

$$f(x) = -x \log x \quad , \text{wh}$$

we have,

$$f\left(\sum_j \lambda_{ij} P(a_j)\right) = -\sum_j \lambda_{ij} P(a_j) \log\left[\sum_k \lambda_{ik} P(a_k)\right]$$

$$= H(A')$$

Since $f(x)$ is concave, which means

$$f\left(\sum_i \alpha_i x_i\right) \geq \sum_i \alpha_i f(x_i)$$

where $\alpha$'s contain the $\lambda$ terms.

Thus $\sum_j \lambda_{ij} f(P(a_i))$

$$= -\sum_j \lambda_{ij} P(a_i) \log P(a_i) = H(A)$$

Therefore $H(A') \geq H(A)$.

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Q 3

a) Using that

$$\rho = \sum_k w_k |\lambda_k \rangle\langle \lambda_k|$$

Then

$$\langle \psi|\rho|\psi\rangle = \sum_k w_k \langle \psi|\lambda_k\rangle\langle\lambda_k|\psi\rangle$$

$$= \sum_k w_k |\langle \psi|\lambda_k\rangle|^2$$

as $w_k \geq 0$ and $|\langle\psi|\lambda_k\rangle|^2 \geq 0$ for all $k$

Then

$$\sum_k w_k |\langle\psi|\lambda_k\rangle|^2 \geq 0$$

$$\Rightarrow \langle\psi|\rho|\psi\rangle \geq 0$$

Hence $\rho$ is a positive operator.

b) It follows that

If $|\psi\rangle = |\rho_m\rangle$, where $|\rho_m\rangle$ is the $m^{th}$ eigenvector with eigenvalue $\rho_m$

$$\langle\rho_m|\rho|\rho_m\rangle = \sum_k \cancel{\delta_k} w_k |\langle\rho_m|\lambda_k\rangle|^2$$

$$\cancel{=} \cancel{w_m} = \rho_m$$

Hence

$$\rho = \sum_m \rho_m |\rho_m\rangle\langle\rho_m|$$

Since $\rho_m = \langle\rho_m|\rho|\rho_m\rangle \geq 0$

$$\rho_m \geq 0$$

So $\rho_m$ is positive and real.

c)
$$\rho(t) = U\rho(0)U^{+}$$

$$\rho^2(t) = [U\rho(0)U^{+}]^2$$
$$= U\rho(0)U^{+}U\rho(0)U^{+}$$
$$= U\rho(0)^2U^{+}$$

Using that $U^{+}U = I$

Then
$$Tr[\rho^2(t)] = Tr[U\rho^2(0)U^{+}]$$

- Using the cyclic property of trace
$$Tr(AB) = Tr(BA)$$

$$\Rightarrow Tr[\rho^2(t)] = Tr[U^{+}U\rho^2(0)]$$
$$= Tr[\rho^2(0)]$$

Hence purity does not change under unitary evolutions.

Q4

a) $r_x$, $r_y$ and $r_z$ are the expectation value of measuring the matrix $\rho$, in the pauli operators $\sigma_x$, $\sigma_y$ and $\sigma_z$ respectively.

- Using the relation

$$r_i = Tr[\rho \sigma_i]$$

$$r_x = Tr\left[\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]$$

$$= Tr\left[\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{2} Tr\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= 2/2 = 1$$

$$r_y = Tr\left[\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right]$$

$$= Tr\left[\frac{1}{2}\begin{bmatrix} i & -i \\ i & -i \end{bmatrix}\right]$$

$$= 0$$

$$r_z = Tr\left[\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right]$$

$$= Tr\left[\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\right]$$

$$= 0$$

$$\vec{r} = [1, 0, 0]$$

b)

$$\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 1 \pm i(-i) \\ 1 \pm i(i) & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 1 \pm 1 \\ 1 \mp 1 & 0 \end{bmatrix}$$

$$Tr[\rho\,\sigma_{\pm}] = \frac{1}{4} Tr\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \pm 1 \\ 1 \mp 1 & 0 \end{bmatrix}\right\}$$

Then

$$Tr[\rho\,\sigma_{+}] = \frac{1}{4} Tr\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{4} Tr\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

$$= \frac{1}{2}$$

Similarly $Tr[\rho\,\sigma_-] = 1/2$

- Alternatively Trace is a linear function

$$Tr[\rho\,\sigma_{\pm}] = Tr\left[\rho\left(\frac{\sigma_x \pm i\sigma_y}{2}\right)\right]$$

$$= \frac{1}{2}\left\{Tr[\rho\,\sigma_x] \pm i\,Tr[\rho\,\sigma_y]\right\}$$

$$= \frac{1}{2}\left\{r_x \pm i\,r_y\right\}$$

c) If $\hat{\rho}$ is pure then $\vec{r}$ lies on the surface of the block sphere : $|\vec{r}|^2 = 1$

$$\hat{\rho}^2 = \frac{1}{4}\left[\hat{I} + \vec{r}.\vec{\sigma}\right]^2$$

$$= \frac{1}{4}\left[\hat{I} + 2\,\vec{r}.\vec{\sigma} + (\vec{r}.\vec{\sigma})^2\right]$$

where

$$(\vec{r}.\vec{\sigma})^2 = \left(r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right)^2$$

$$= r_x^2\sigma_x^2 + r_xr_y\sigma_x\sigma_y + r_xr_z\sigma_x\sigma_z$$

$$+ r_yr_x\sigma_y\sigma_x + r_y^2\sigma_y^2 + r_yr_z\sigma_y\sigma_z$$

$$+ r_zr_x\sigma_z.\sigma_x + r_zr_y\sigma_z\sigma_y + r_z^2\sigma_z^2$$

$$= \left[r_x^2 + r_y^2 + r_z^2\right]I + r_xr_y\left[\sigma_x, \sigma_y\right]$$

$$+ r_yr_z\left[\sigma_y, \sigma_z\right] + r_zr_x\left[\sigma_z, \sigma_x\right]$$

$$= |r|^2 I + 2i\left[r_xr_y\sigma_z + r_yr_z\sigma_x + r_zr_x\sigma_y\right]$$

$$\Rightarrow \hat{\rho}^2 = \frac{1}{4}\left[I(1+|r|^2) + 2\left[r_x + ir_yr_z, r_y + ir_zr_x, r_z + ir_xr_y\right].\vec{\sigma}\right]$$

$$= \frac{1}{4}\left[I(1+|r|^2) + 2\,\vec{r}'.\vec{\sigma}\right]$$

So For pure state

$$\hat{\rho}^2 = \frac{1}{2}\left[I + \vec{r}'.\vec{\sigma}\right]$$

$$Tr\left[\hat{\rho}^2\right] = \frac{1}{2}Tr\left[I\right]$$

$$= 1$$

Q5

$$\hat{\rho} = (1-\varepsilon)|00\rangle\langle 00| + \varepsilon |\phi^+\rangle\langle\phi^+|$$

where $|\phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$

$$\hat{\rho} = \begin{bmatrix} 1-\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \varepsilon/2 & 0 & 0 & \varepsilon/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon/2 & 0 & 0 & \varepsilon/2 \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 1-\varepsilon/2 & 0 & 0 & \varepsilon/2 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \varepsilon/2 & 0 & 0 & \varepsilon/2 \end{array}\right]$$

$$\hat{\rho}^{PT} = \begin{bmatrix} 1-\varepsilon/2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon/2 & 0 \\ 0 & \varepsilon/2 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon/2 \end{bmatrix}$$

Eigenvalues of $\hat{\rho}^{PT}$ : $\left\{ 1-\varepsilon/2,\ \varepsilon/2,\ \varepsilon/2,\ -\varepsilon/2 \right\}$

One eigenvalue is always negative hence this state is always entangled.

Q6

$$\hat{\rho} = \frac{(1-\varepsilon)}{4}\,\hat{I} + \varepsilon|\psi^-\rangle\langle\psi^-|$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$

$$\hat{\rho} = \frac{1-\varepsilon}{4}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} + \frac{\varepsilon}{2}\begin{bmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{bmatrix}$$

$$= \begin{bmatrix}(1-\varepsilon)/4 & 0 & 0 & 0\\ 0 & (1+\varepsilon)/4 & -\varepsilon/2 & 0\\ 0 & -\varepsilon/2 & (1+\varepsilon)/4 & 0\\ 0 & 0 & 0 & (1-\varepsilon)/4\end{bmatrix}$$

$$Tr[\hat{\rho}^2] = Tr[\ \dots\ ]$$

$$= \left[\frac{(1-\varepsilon)}{4}\right]^2 + \left[\frac{1+\varepsilon}{4}\right]^2\left[-\frac{\varepsilon}{2}\right]^2 + \left[-\frac{\varepsilon}{2}\right]^2\left[\frac{1+\varepsilon}{4}\right]^2 + \left[\frac{1-\varepsilon}{4}\right]^2$$

$$= \frac{2}{16}(1-\varepsilon)^2 + \frac{2\varepsilon^2}{4(16)}(1+\varepsilon)^2$$

$$= \frac{1}{32}\left[\varepsilon^4 + 3\varepsilon^3 + 5\varepsilon^2 - 8\varepsilon + 4\right]$$

For $\hat{\rho}$ to be pure

$$Tr[\hat{\rho}^2] = 1$$

$$\varepsilon = \{1.77,\ -2,\ -1.38 \pm 2.44i\}$$

Hence $\hat{\rho}$ is not pure.

Q7

a) Using the Schrodinger Equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

$$\frac{\partial}{\partial t} |\psi\rangle = -\frac{i\hat{H}}{\hbar} |\psi\rangle$$

Then

$$\frac{d}{dt}\hat{\rho} = \frac{d}{dt}\left[\sum_k w_k |\psi_k\rangle\langle\psi_k|\right]$$

$$= \sum_k \left(\frac{\partial w_k}{\partial t}\right)|\psi_k\rangle\langle\psi_k|$$

$$+ \sum_k w_k \left\{\left[\frac{\partial |\psi_k\rangle}{\partial t}\right]\langle\psi_k| + |\psi_k\rangle\left[\frac{\partial\langle\psi_k|}{\partial t}\right]\right\}$$

* Assumption, the states $|\psi_k\rangle$ evolve so $\frac{\partial w_k}{\partial t} = 0$

$$= \sum_k w_k \left\{\frac{-i\hat{H}}{\hbar}|\psi_k\rangle\langle\psi_k| + \frac{i}{\hbar}|\psi_k\rangle\langle\psi_k|\hat{H}\right\}$$

$$= \frac{-i\hat{H}}{\hbar}\hat{\rho} + \cancel{\frac{i\hat{H}\hat{\rho}}{\hbar}} + \frac{i\hat{\rho}\hat{H}}{\hbar}$$

$$= \frac{-i}{\hbar}\left[\hat{H}\hat{\rho} - \hat{\rho}\hat{H}\right]$$

$$\frac{d\hat{\rho}}{\partial t} = \frac{-i}{\hbar}\left[\hat{H}, \hat{\rho}\right]$$

b) $U = \exp\left[-i H t/\hbar\right]$

$= \exp\left[-i\left[\frac{\hbar w \sigma_z}{2}\right]\frac{t}{\hbar}\right]$

$= \exp\left[-i w \sigma_z t/2\right]$

$U^\dagger = \exp\left[i w \sigma_z t/2\right]$

$\hat{\rho}(0) = \frac{\hat{I} + \sigma_x}{2}$

Then

$\hat{\rho}(t) = U \rho U^\dagger$

$= \frac{1}{2}\left[U(\hat{I} + \sigma_x) U^\dagger\right]$

$= \frac{1}{2}\left[U U^\dagger + U \sigma_x U^\dagger\right]$

$= \frac{1}{2}\left[\hat{I} + U \sigma_x U^\dagger\right]$

where $U \sigma_x U^\dagger = e^{-iwt/2\,\sigma_z}\,\sigma_x\, e^{iwt/2\,\sigma_z}$

* Using the relation $e^{-i\theta C} A e^{i\theta C} = A\cos(\theta) + B\sin(\theta)$

showed in class, where: $[A, B] = iC$

$\Rightarrow \quad [\sigma_x, \sigma_y] = 2i\sigma_z$ , let $C = 2\sigma_z$ , $\theta = \frac{wt}{4}$

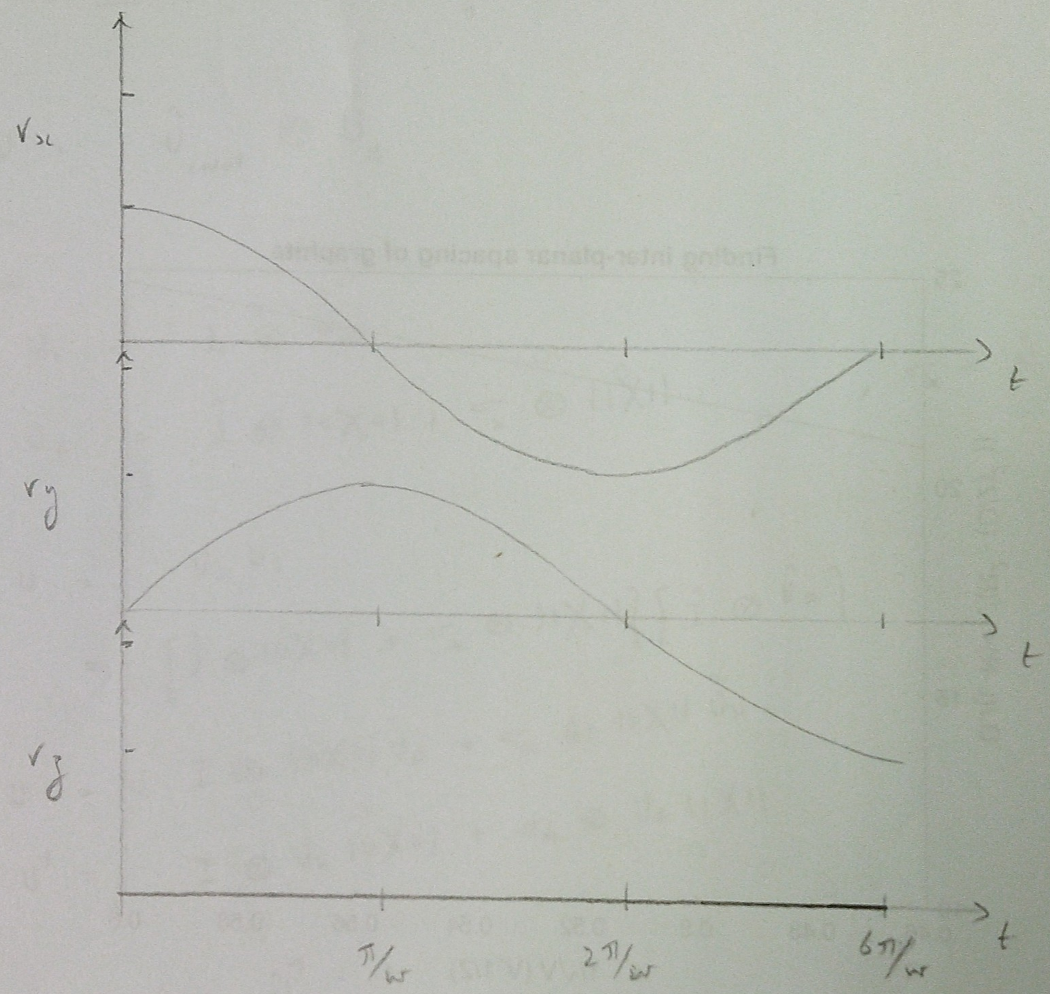
$U \sigma_x U^\dagger = e^{-i\theta C} A e^{i\theta C}$

$= \sigma_x \cos(wt/4) + \sigma_y \sin(wt/4)$

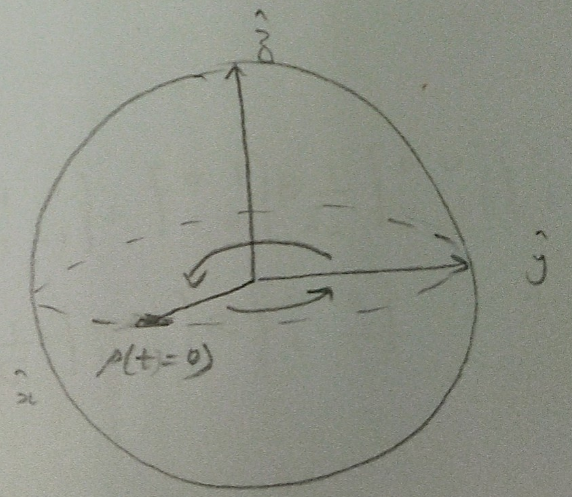
Then $\hat{\rho}(t) = \frac{1}{2}\left[\hat{I} + \langle \cos(wt/4), \sin(wt/4), 0\rangle \vec{\sigma}\right]$

c) $\vec{r}(t) = \langle \cos(\omega t/\hbar), \sin(\omega t/\hbar), 0 \rangle$

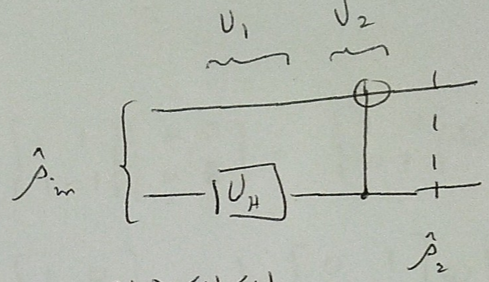
$\vec{r}(t)$ against $t$




On the Bloch sphere,



Q8




$$\hat{\rho}_{in} = \frac{3}{4}\,|0\rangle|0\rangle\langle 0|\langle 0| + \frac{1}{4}\,|1\rangle|1\rangle\langle 1|\langle 1|$$

$$U = \hat{U}_{cNot} \otimes \hat{U}_H$$

where

$$U_1 = \hat{I} \otimes \hat{U}_H$$

$$U_2 = \hat{I} \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1| \quad , \sigma_x \equiv \hat{N}$$

$$U = U_2 U_1$$

$$= \left[\hat{I} \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1|\right]\left[\hat{I} \otimes \hat{U}_H\right]$$

$$\Rightarrow U = I \otimes |0\rangle\langle 0|\hat{U}_H + \sigma_x \otimes |1\rangle\langle 1|\hat{U}_H$$

$$U^\dagger = I \otimes U_H|0\rangle\langle 0| + \sigma_x \otimes U_H|1\rangle\langle 1|$$

$$\circ \quad \hat{\rho}_2 = \hat{U}\hat{\rho}_{in}\hat{U}^\dagger$$

* Using the matrix representation

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

* Similarly

$$U^{+} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

- Checking

$$UU^{+} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \hat{I}$$

Then $\hat{\rho}_2 = U \hat{\rho}_{in} U^{+}$

$$= \frac{1}{4}\left(\frac{1}{2}\right) \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \hat{\rho}_2 = \frac{1}{8}\begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix}$$

* Taking partial trace to find the reduced matrix

$$\rho_B = Tr_A [\hat{\rho}_2]$$

$$= \frac{1}{8}\begin{bmatrix} 3+1 & 0+0 \\ 0+0 & 1+3 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{\hat{I}}{2}$$