Introduction to Quantum Information Science and Quantum Technologies

Assignment 4

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"What a fine day for science." - *Dexter*

Question 1

(a) Prove using the L'Hopital's rule that

$$\lim_{x \to 0} xln(x).$$

(b) Prove from the first principle definition that the conditional entropy of event B given A is

$$H(B|A) = H(A, B) - H(A).$$

Question 2

The Shannon entropy is given by

$$H(A) = -\sum_i p(a_i) logp(a_i).$$

(a) Show, using the Lagrange multiplier method, that entropy is maximized when $p(a_i) = 1/d$.

(b) If the probabilities are replaced by

$$p(a_i^{'}) = \sum_j \lambda_{ij} p(a_i), \qquad \lambda_{ij} \in \{0, 1\}$$

use the concavity of the function $f(x) = -xlog(x)$ to show that

$$H(A') \geq H(A).$$

Question 3

(a) Show that $\hat{\rho}$ is a positive operator which means that $\langle\psi|\,\hat{\rho}\,|\psi\rangle \geq 0$ for any $|\psi\rangle$.

(b) Show that its eigenvalues ρ_m are real and positive. Hence

$$\rho = \sum_m \rho_m \,|\rho_m\rangle\,\langle\rho_m|$$

where $|\rho_m\rangle$ are the corresponding eigenvectors.

(c) How does the purity of a quantum state change as it evolves under a unitary operation?

Question 4

Any density matrix can be written as,

$$\hat{\rho} = \frac{1}{2}(\hat{I} + \overrightarrow{r}\cdot\overrightarrow{\sigma})$$

where $\overrightarrow{r} = \{r_x, r_y, r_z\}$ and $\overrightarrow{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$.

(a) Find $\overrightarrow{r}$ given that $\hat{\rho} = |+\rangle\,\langle+|$ using the trace operator. Where the state $|+\rangle$ is defined as,

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

(b) Derive the expectation values of $\hat{\rho}$ for the operators $\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

(c) Show that if $\hat{\rho}$ is a pure state, the trace of $\hat{\rho}^2$ is always unity.

Question 5

For what value of ε will the state

$$\hat{\rho} = (1-\varepsilon)\,|00\rangle\,\langle 00| + \varepsilon\,\left|\phi^+\right\rangle\left\langle\phi^+\right|$$

be entangled? Use the positivity of the partial transpose test. Here $|\phi^+\rangle$ is bell state.

Question 6

What is the purity of the Weiner state defined as

$$\hat{\rho} = (1-\varepsilon)\frac{\hat{I}}{4} + \varepsilon\left|\psi^-\right\rangle\left\langle\psi^-\right|$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is the singlet state?

Question 7

(a) Show that for the density matrix formalism, the density matrix evolves under a time-independent Hamiltonian as per the Liouville–von Neumann equation,

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}].$$

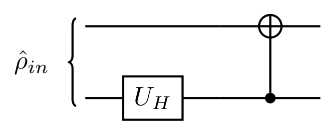
(b) Hence solve for $\hat{\rho}(t)$ given,

$$\hat{\rho}(0) = \frac{\hat{I} + \sigma_x}{2},$$
$$\hat{H} = \frac{\hbar w \sigma_z}{2}.$$

(c) Plot $\overrightarrow{r}$ against time for $\hat{\rho}(t)$ and represent this evolution on the Bloch sphere.

Question 8

Consider the following circuit with $\hat{\rho}_{in} = 3/4\,|00\rangle\langle 00| + 1/4\,|11\rangle\langle 11|$.



Find the density matrix after the Bell creation circuit. What is the reduced matrix for the second qubit? (Trace out the first qubit).