HHL Algorithm for Linear Systems of Equations

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Abstract

The HHL algorithm, proposed by Aram Harrow, Avinatan Hassidim, and Seth Lloyd in 2009, is used for solving linear systems of equations. We compare the operation counts of the classical algorithms with the HHL algorithm which is a quantum algorithm that offers an exponential boost to computation speed. To solve such a linear system, we cast our problem in the form $A|x\rangle = |b\rangle$, where $|x\rangle$ and $|b\rangle$ are normalized vectors and A is a hermitian matrix. The process involves finding the eigenvalues of the matrix by making use of the Quantum Phase Estimation (QPE) sub-routine. This in turn makes use of the inverse Quantum Fourier Transform (QFT). The determined eigenvalues are then used to implement a controlled rotation to effectively find the inverse of the matrix A. This allows us to calculate $|x\rangle = A^{-1}|b\rangle$. The final step is to uncompute the phase estimation. We next discuss the implementation of this algorithm on physical hardware and simulate the results on IBM's quantum computer.

1 Introduction

Linear systems of equations are widespread in many sub-fields of physics, engineering, mathematics, finance, and computer science. Classically, the best algorithms for solving an $N \times N$ system, such as Gaussian elimination which incorporates pivoting, take polynomial time, i.e., $O(N^3)$ [2]. But with the progress of these fields, we encounter ever larger systems of equations where even polynomial time is too costly. This is where the strength of quantum computing shines, which can offer exponential speed ups. The HHL algorithm is one such implementation of quantum computing principles. Introduced in 2009 by Harrow, Hassidim and Lloyd, it can be used to solve the matrix equation $A|x\rangle = |b\rangle$, where A is a hermitian matrix. This quantum advantage, however, comes at the cost of detail in the output information. To wit, the algorithm does not provide the complete solution vector. Instead, it generates a function of the solution vector, which would still enable the user to extract useful information about the solution vector. This can be thought of as finding the expectation value of an operator M acting on the solution vector as $\langle x|M|x\rangle$ and measuring the outcome [1, 4].

This algorithm serves as a subroutine of more advanced algorithms such as those pertaining to Machine Learning and the modeling of quantum systems [2]. Indeed, with advances in quantum computing it may gain further importance as its limitations are overcome.

2 Mathematical Formulation

2.1 Formulating the problem

Our problem is expressed as:

$$A|x\rangle = |b\rangle, \tag{1}$$

where A is an $N \times N$ hermitian matrix, and $|x\rangle$ and $|b\rangle$ are normalized vectors. Using the spectral decomposition theorem, we can express A as:

$$A = \sum_{j=0}^{N-1} \lambda_j |u_j\rangle\langle u_j|, \tag{2}$$

Since A is hermitian, its eigenbasis is orthonormal. We can also calculate the inverse of A as:

$$A^{-1} = \sum_{j=0}^{N-1} \lambda_j^{-1} |u_j\rangle\langle u_j|. \tag{3}$$

Further, we express $|b\rangle$ in the eigenbasis of A as:

$$|b\rangle = \sum_{i=0}^{N-1} b_i |u_i\rangle. \tag{4}$$

Now, the solution vector for eq. (1) is given by:

$$|x\rangle = A^{-1}|b\rangle.$$

By substituting in equations (3) and (4) into the above, we get:

$$\begin{aligned} |x\rangle &= \left(\sum_{j=0}^{N-1} \lambda_j^{-1} |u_j\rangle\langle u_j| \right) \left(\sum_{i=0}^{N-1} b_i |u_i\rangle \right) \\ &= \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} \lambda_j^{-1} b_i |u_j\rangle\langle u_j|u_i\rangle. \end{aligned}$$

Hence:

$$|x\rangle = \sum_{j=0}^{N-1} \lambda_j^{-1} b_j |u_j\rangle. \tag{5}$$

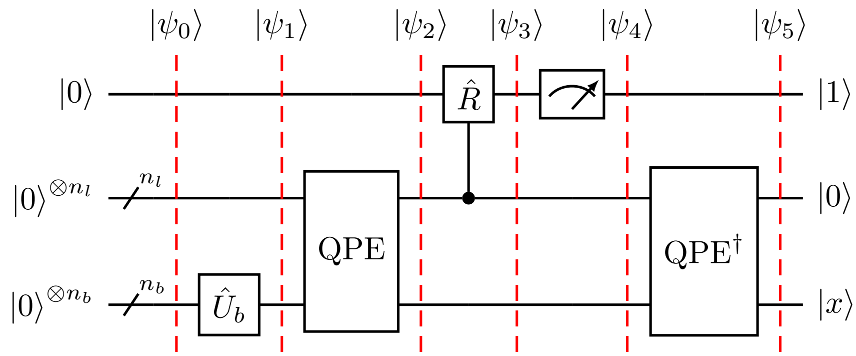

Figure 1: Circuit diagram.

2.2 Tracing the evolution of the state

We can trace the evolution of the state in a step-wise fashion with reference to Fig. 1: the circuit contains three registers. The first register is based on n_b qubits, so that it can be encoded with the value of $|b\rangle$. The size of the $|b\rangle$ vector is thus $N \times 1$ where $N = 2^{n_b}$. The second register, also known as the clock register, is based on n_l qubits. This is meant to hold the eigenvalues of A. The final register is a single ancilla qubit. In the diagram, the convention followed is that the most significant bit is at the bottom and the least significant bit is at the top.

We initialize the state as [2]:

$$|\psi_0\rangle = |0\rangle^{\otimes n_b}|0\rangle^{\otimes n_l}|0\rangle.$$

We encode the information for $|b\rangle$ onto the n_b register using a unitary operation $\hat{U}_b$ such that $\hat{U}_b|0\rangle = |b\rangle$ to obtain the following:

$$|\psi_1\rangle = |b\rangle^{\otimes n_b}|0\rangle^{\otimes n_l}|0\rangle.$$

Next, we apply the Quantum Phase Estimation subroutine to encode the eigenvalues on to the n_l register. This results in the following state:

$$|\psi_2\rangle = |b\rangle^{\otimes n_b}|\lambda_j\rangle^{\otimes n_l}|0\rangle.$$

We now want to find the inverse of the eigenvalues. To this end, we apply a controlled rotation on the ancilla qubit in the top register, based on the value of the n_l register [3]. Thus we obtain following state:

$$|\psi_3\rangle = \sum_{j=0}^{N-1} b_j|u_j\rangle|\lambda_j\rangle\left(\sqrt{1-\frac{C^2}{\lambda_j^2}}|0\rangle + \frac{C}{\lambda_j}|1\rangle\right).$$

In the above equation, C is a normalization constant for the third register. The next step is to perform a measurement on the third register. The desired outcome is $|1\rangle$, which will result in the following state:

$$|\psi_4\rangle = \sum_{j=0}^{N-1} b_j |u_j\rangle |\lambda_j\rangle \left(\frac{C}{\lambda_j} |1\rangle \right).$$

If the measurement had resulted in the $|0\rangle$ state, we would have repeated the procedure until we obtained the desired state. With a little manipulation, the state $|\psi_4\rangle$ becomes:

$$|\psi_4\rangle = C \sum_{j=0}^{N-1} b_j \lambda_j^{-1} |u_j\rangle |\lambda_j\rangle |1\rangle.$$

We then enter the uncompute stage, where we apply the inverse phase estimation, hence returning the n_l register to the $|0\rangle$ state.

$$|\psi_5\rangle = C \sum_{j=0}^{N-1} b_j \lambda_j^{-1} |u_j\rangle |0\rangle^{\otimes n_l} |1\rangle.$$

Now we know from equation (5) that $|x\rangle = \sum_{j=0}^{N-1} \lambda_j^{-1} b_j |u_j\rangle$, so

$$|\psi_5\rangle = C \sum_{j=0}^{N-1} |x\rangle |0\rangle^{\otimes n_l} |1\rangle.$$

We can see that the final state $|\psi_5\rangle$ is proportional to our solution vector.

References

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