Report Draft

Salman Shahid and Hammas Ali

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1 Abstract

A trip through the world of classical and and quantum information theory is to be undertaken. The notions of information, entropy, and probability will be studied along with their deep and intimate connections. Important quantities in quantum information theory will be studied along with their properties. These quantities include the entanglement entropy, average entropy, relative entropy, the Pinsker inequality, mutual information, and correlations. During the trip, the aforementioned concepts shall be applied to a few quantum mechanical systems.

2 Introduction

Information and probability are intricately related, and studying this deep connection between the two is of utmost importance in the field of quantum information science. This connection helps us incorporate another fundamental property of the world we inhabit: entropy. The aforementioned notions can be studied in the context of the vastest to the smallest. They can help us explore the bangs of black holes and help us fathom the whispers of atoms. Information, to put it simply, is how much one can possibly learn (or be surprised) about a system, and probability quantifies the chances of an event occurring. Let us try to illuminate the connection between the two using an example. The probability that it will be raining when I go to PDC is 0.1. However, if I open my window to check if it is raining currently, I have gained information about the weather. This means that the probability that it will be raining when I leave for PDC has changed. The change depends on what I saw when I opened my window. Trying to gain information about the system has led to a change in the probability of an event. Now, we must also recognize the connection between information and entropy. It shall serve us well to see this connection using another example. Consider an event A that has a_i as the possible outcomes. Let's say if the probability of the event a_3 happening is 1, and we measure the system and find it to be in a_3. Have we acquired information about the system? Well, not at all. However, if the probability of the event a_3 happening is 0.2 and we measure the system to be in a_3, we have gained significant information about the system because there was 0.8 probability of events other than a_3 happening. Let us take inspiration from this example and find an expression that connects information and entropy in an elegant form. How much information will we gain if we learn that the system is in a_3 and b_1? Where b_1 is now another outcome from another event. Well, the information we gain should be additive while the probability of both events is multiplicative. The logarithm function is additive when its input is a product. This allows us to motivate the following form for information,

$$-K \sum_i P(a_i) \log P(a_i) \tag{1}$$

Where K is some constant. We shall present an examination of classical information theory, and shall also explore how the notions of information theory extend to quantum mechanical systems. More specifically, we shall be studying the notions of entanglement, entropy, relative entropy, and mutual information in the context of different quantum systems. We will be making several claims about the all aforementioned quantities.

3 Mathematical Formulation

Let us now quantify the foregoing. Consider a bipartite system made up of the subsystems A and B. All our discussion concerns pure quantum states, $|\psi\rangle$. Quantifying the degree of entanglement of this state tantamounts to quantifying how much one can learn about one system by doing a measurement on the other. One way to do so is to measure the entanglement entropy (also known as the Von Neumann entropy) of either subsystem. This is defined as

$$S_A(|\psi\rangle) = -\operatorname{Tr}_A\left(\rho_A \log\left(\rho_A\right)\right) \tag{2}$$

where ρ_A is the reduced density matrix of A. An essential result states that the entropy of both subsystems is the same for any state $|\psi\rangle$. This result supports our definition for the entanglement entropy.

$$S_A(|\psi\rangle) = S_B(|\psi\rangle) \tag{3}$$

Interestingly, this function is bounded from above and below in the following way,

$$0 \leq S_A(|\psi\rangle) \leq \min(\log(d_A), \log(d_B)) \tag{4}$$

Here d_A and d_B are the dimensions of A and $B's$ Hilbert space respectively. While the lower bound may seem obvious, the upper bound is decidedly not so. Entanglement entropy serves as the lower bound of another type of entropy: the measurement entropy.

$$S_A(|\psi\rangle) \leq S(|\psi\rangle, O_A) \tag{5}$$

Here O_A is any observable of system A and $S(|\psi\rangle, O_A)$ is the entropy of measurements in A. This latter notion quantifies, in essence, the uncertainty in the measurement outcomes of O_A in the following way,

$$S(|\psi\rangle), O_A) = \sum_{n=1}^{\mathrm{d_A}} -P_n \log\left(P_n\right) \tag{6}$$

Here P_n is the probability that observing O_A yields the eigenvalue o_n. (6) claims that the entanglement entropy serves as a lower bound for the the entropy of measurement for any observable on A. Conversely, and perhaps more interestingly, if there exists any observable on A the produces a certain result then A and B are not entangled.

To demonstrate these notions , we first lay out two ways to produce entangled states. Let us consider spin-1/2 systems. One route to go about this is to create the ground state of the Hamiltonian

$$\left(\vec{J}_A + \vec{J}_B\right)^2 \tag{7}$$

where $\vec{J}_A$ is the vector spin operator for system A. The second route is to start with an unentangled state and produce one through time evolution under a Hamiltonian that has A and B interacting, like the Hamiltonian cited above. In our concrete example, the entanglement entropy varies with time. The calculations and graphs will be provided later. The foregoing raises a question: what is the average entropy of the state as it evolves? Our states evolves into different parts of the Hilbert Space and the average calculated is over this collection of states. This is the average entropy over time. However, there is another average that merits discussion, especially vis-a-vis the former. We consider an ensemble of random states spread all over the Hilbert space and calculate their average. This is the average entropy over an ensemble of states.

Another important quantity that shall help us differentiate between two quantum systems is the relative entropy. This is defined in the following way,

$$S(\rho \big| \sigma) = tr(\rho \log \rho - \rho \log \sigma) \tag{8}$$

ρ and σ are density matrices. As an example, let us consider the relative entropy of the two states, ρ_{AB} and $\rho_A \otimes \rho_B$. ρ_{AB} is the density matrix of a bipartition with subsystems, A and B and $\rho_A \otimes \rho_B$ is the tensor product of two reduced density matrices. It can be shown that the relative entropy quantity boils down to the following,

$$S(\rho_{AB}|\rho_A \otimes \rho_B) = S_A + S_B - S_{AB} \tag{9}$$

Where S_A is the entanglement entropy of the subsystem A and S_B is the entanglement entropy of the subsystem B. S_{AB} is the entropy of the bipartition.

Fascinatingly, it is bounded below in the following way,

$$S(\rho\big|\sigma) \geq 0 \tag{10}$$

(10) makes sense because if the density matrices are the same, the relative entropy must become zero. Relative entropy can serve as an upper bound of another important quantity. This inequality takes the following form,

$$\frac{1}{2||O||}(tr(O\rho) - tr(O\sigma))^2 \leq S(\rho\big|\sigma) \tag{11}$$

where σ and ρ are density operators. (11) is true for all bounded observables O. $||O||$ is the norm of a bounded operator that is defined as the following sense,

$$||O|| = \sup_{|\psi\rangle} \sqrt{\frac{\langle\psi| O^\dagger O |\psi\rangle}{\langle\psi|\psi\rangle}} \tag{12}$$

Where sup is the supremum over the states $|\psi\rangle$. Let us also mention a few other results that are of extreme importance in our quest and can help us prove (11). The first result is the following

$$S(\rho\big|\sigma) \geq \frac{1}{2}(||\rho - \sigma||_1)^2 \tag{13}$$

(13) is known as the Pinsker inequality. The second result is the following

$$||X||_1 \geq tr\bigg(X\frac{O}{||O||}\bigg) \tag{14}$$

(14) must also hold true for all observables O. For both (13) and (14), we employ the trace norm. The trace norms are defined in the following manner,

$$\begin{aligned} ||X||_1 &= tr\bigg(\sqrt{X^\dagger X}\bigg) \\ ||\rho - \sigma||_1 &= tr\bigg(\sqrt{(\rho - \sigma)^\dagger(\rho - \sigma)}\bigg) \end{aligned} \tag{15}$$

Where the square root is of a positive operator and is itself positive . Let us define another important quantity, mutual information, in the following manner,

$$I_{ABC} = S_A + S_B - S_{AB} \tag{16}$$

Where S_A, S_B, and S_{AB} are defined just like they were in (9). I_{ABC} is bounded below in the following way,

$$I_{ABC} \geq 0 \tag{17}$$

We can also show that mutual information provides an upper bound on correlations between the subsystems A and B in the following manner,

$$\frac{1}{2}\bigg(\frac{\langle\psi| O_A O_B |\psi\rangle - \langle\psi| O_A |\psi\rangle \langle\psi| O_B |\psi\rangle}{||O_A||\,||O_B||}\bigg)^2 \leq I_{ABC} \tag{18}$$

(18) holds true for all observables in A, O_A, and all observables in B, O_B.

4 Conclusion

The display of equations above has been stupendous. However, a more intuitive interpretation of these equations is missing. Alas, the proofs and plots too are omitted. Fret not, my dear seeker of knowledge, the documents that shall be submitted later shall start from classical information theory and extend those notions to various quantum mechanical systems. We shall also soon satiate your yearnings with proofs, more interesting theorems and plots. Soon we will remove the mysterious veil over the universe.

5 Bibliography

- Quantum Information Theory, Mark M. Wilde
- Quantum Information Stephen, M. Barnett
- Quantum Information for Quantum Gravity: Selected Topics - Lecture 1 - Eugenio Bianchi
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