PHY 612 / CS 5112 / EE 539
Introduction to Quantum Information Science and Quantum Technologies

Final Exam

7 May 2023 TIME ALLOWED: 1.5 HOURS

"OOGA BOOGA BOOGA!" - *Eustace*

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains five (5) questions.
2. Answer all questions. The marks for each question are indicated at the beginning of each question.
3. Add your ID and name to the answer booklets and label the booklets properly if you use more than one.
4. This exam is open class notes. You may not consult your assignments, book, or any other resources, online or otherwise.
5. Please write down systematically the steps in the workings.

Question 1. (10 marks)
Consider the linear entropy

$$L = \sum_{i=1}^{d} p_i(1 - p_i). \tag{1}$$

Using the method of Lagrange multipliers, shows that L is maximized when all the p_i's are equal. What is the maximal linear entropy?

Question 2. (10 marks)
A one-qubit state can be expressed in terms of a Bloch (also called a polarization) vector

$$\rho = \frac{1}{2}\big(\mathbf{1} + \overrightarrow{p} \cdot \overrightarrow{\sigma}\big) \tag{2}$$

where $\overrightarrow{\sigma} = \big(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\big)$ is a vector whose elements are the Pauli operators. The length of the Bloch vector is defined as $p = \|\overrightarrow{p}\| = \sqrt{p_x^2 + p_y^2 + p_z^2}$. Express this length in terms of the state purity.

Question 3. (10 marks)
What is the Von-Neumann entropy of the Wener state

$$\rho = (1 - \varepsilon)\frac{\mathbf{1}}{4} + \varepsilon \left|\psi^-\right\rangle \left\langle\psi^-\right|, \tag{3}$$

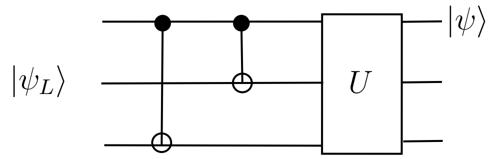
$|\psi^-\rangle$ being one of the four Bell states?

Question 4. (10 marks)
Fault tolerant quantum circuits automatically correct for errors of some kind. Suppose we have a qubit in the state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ which we wish to transmit from Alice to Bob. However, the transmission channel corrupts this qubit by flipping the bit $0 \leftrightarrow 1$. One method to protect the qubit from bit-flips is to encode this single qubit into three qubits in the form of a logical qubit $|\psi_L\rangle = a\,|000\rangle + b\,|111\rangle$. We consider only bit flips on a *single* qubit.

This logical (=physical three-qubit) system goes over a transmission channel which can corrupt this to:

$$a\,|100\rangle + b\,|011\rangle\,, \tag{4}$$
$$a\,|010\rangle + b\,|101\rangle\,, \text{or} \tag{5}$$
$$a\,|001\rangle + b\,|110\rangle\,. \tag{6}$$

Build a fault tolerant circuit that automatically provides the desired $|\psi\rangle$ at its output on one of the channels, no matter which single bit is flipped. HINT: What $\hat{U}$ is needed in the given circuit?



Question 5. (10 marks)

A QKD protocol is based on the sharing of an entangled pair of qubits. Suppose Alice and Bob share a Bell pair,

$$\left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), \tag{7}$$

where Alice gets the first qubit and Bob the second. They can make measurements in their respective bases, announce what bases they chose, and use as key the measurement outcomes when the bases match. The outcomes will ideally be perfectly correlated. Argue your way through this.

Now if the clever Eve takes over the entanglement generation part and produces a maximally mixed state

$$\rho = \frac{1}{2}\,|00\rangle\,\langle 00| + \frac{1}{2}\,|11\rangle\,\langle 11|\,, \tag{8}$$

and Alice and Bob were to repeat their algorithm, their outputs may still be correlated. Discuss how can Alice and Bob discriminate whether they are getting genuine entangled qubits or those from a mixed state. What can kind of expectation values would they need to measure?

END OF PAPER