Quantum Image Representation- FRQI Image

Gohar Raiz Haider (22120005) and Wajeeha Rizwan (22120001)
Physics Department, School of Science and Engineering, LUMS.
(Dated: May 1, 2023)

The world of quantum imaging was explored in this work starting with image representation in the quantum realm. FRQI (flexible representation of quantum images) allow encoding of classical square images into a normalized quantum state via color and pixel values. The quantum circuit for FRQI was run on IBM interface and measurement yields a probability distribution. Afterwards efforts were made in order to reconstruct the quantum state FRQI using a brute force approach using the probabilities of detected states in zeeman basis. An attempt was also made to recover the classical image through reverse engineering the encoding process. Finally evaluation metrics including fidelity and meas square error were calculated and they turned out to be 0.99 and 200 respectively.

I. INTRODUCTION

Although a relatively new concept in the field of science and technology, quantum computing is being researched on and used extensively for finding solutions to problems that seem too complicated for the existing classical computers. Though the quantum computers are quite different and rather complex to use but the promise of exponentially increasing the speed and efficiency of problem solving has created an interest among researchers worldwide to dig more into the applications of this field. Quantum computers are basically even more advanced than the supercomputers. Even those problems that seem extremely complex for supercomputers, like the modelling of atoms in a compound, quantum computers can perform such tasks rather easily. Currently, quantum computers and quantum technology in general is being used in various applications like electric vehicles, solving of complex energy challenges, quest of solving space and cosmic mysteries, image processing and a variety of a lot of other applications [1].

One of the most interesting area where Quantum science and technology is being utilized is Quantum Image Processing. With a multitude of advances in technology, image processing has become an extensively researched upon area of technology. It is being employed in various different disciplines and areas of research. Facial recognition, automated vehicles, image Photoshop and numerous other techniques use image processing as their basis. Utilizing Quantum technology in image processing is what Quantum image processing is all about. As is evident until now, quantum image processing will be a lot more efficient in terms of speed and time than its classical counterpart. It will also prove to be extremely useful for simple day to day applications like simple face recognition on a mobile phone or criminal detection at a police station, but all of this might be possible in a fraction of time and with much less error than is now using classical techniques. Image compression, edge detection, image storage, image retrieval, binary image line detection are just some of the tasks achievable by quantum image processing [2, 3]. In order to carry out Quantum Image Processing, the image must first be transformed into its quantum counterpart known as the quantum image. This state can be achieved by a number of different processes. Here we discusses the FRQI technique in detail. The FRQI state represents the classical images, after a transformation, as quantum images on a quantum computer in a normalized state. This state carries the information about the colours in an image and their respective positions. It is quite an efficient method for the preparation of an image on which various different quantum image processing algorithms can be applied to achieve desired results. Not only does FRQI provide a representation to images but can also prove to be extremely useful for the exploration of other tasks performed by the quantum computers regarding image processing [2, 4]. Preparing this state requires a polynomial number of simple operations and gates [3, 4]. We work with a 2x2 image i.e. 4 pixels image, where the color and its position is encoded in the FRQI state as shown below,

$$|I(\theta)\rangle = \frac{1}{2^n}\sum_{i=0}^{2^{2n}-1}(\cos(\theta_i)|0\rangle + \sin(\theta_i)|1\rangle)\otimes|i\rangle, \quad (1)$$

$$\theta_i \in [0, \frac{\pi}{2}], i = 0, 1, 2, ..., 2^{2n} - 1. \quad (2)$$

II. MATHEMATICAL FORMULATIONS

The FRQI state contains coded information in the form of colour and its related pixel position as shown below. FRQI state is prepared through a unitary transformation which has two steps. First applying the hadamard transform $\mathcal{H} = I \otimes H^{\otimes 2}$, where I is the $2D$ identity matrix and H is the hadamard gate, on $|0\rangle^{\otimes 3}$, producing the state $|H\rangle$,

$$(I \otimes H^{\otimes 2})|0^{\otimes 3}\rangle = \frac{1}{2}|0\rangle \otimes \sum_{i=0}^{3}|i\rangle = |H\rangle. \quad (3)$$

State $|0\rangle$ is initialised on all three qubits and hadamard gate is applied on the first two, creating superposition.

The third qubit is our ancillary qubit. In the second step, controlled rotations are applied on the $|H\rangle$ state as defined by,

$$R_i = (I \otimes \sum_{j=0, j\neq i}^{3} |j\rangle\langle j|) + R_y(2\theta_i) \otimes |i\rangle\langle i|. \quad (4)$$

where

$$R_y(2\theta_i) = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{pmatrix}$$

The controlled rotations are applied in succession corresponding to the number of pixels, which in our case is 4. This corresponds to a unitary operation $\mathcal{R}$ defined as,

$$\mathcal{R}|H\rangle = \prod_{i=0}^{3} R_i|H\rangle. \quad (5)$$

Equation. I is our state obtained after applying the hadamard transform. Now, the controlled rotation operators are applied in succession as follows.

$$R_0|H\rangle = \left(I \otimes \sum_{i=0, i\neq 0}^{3} |i\rangle\langle i|) + R_y(2\theta_0) \otimes |0\rangle\langle 0| \right) (\frac{1}{2}|0\rangle \otimes \sum_{i=0}^{3} |i\rangle)$$

$$= \frac{1}{2}\left([|0\rangle \otimes \sum_{i=0, i\neq 0}^{3} |i\rangle\langle i| + (\cos(\theta_0)|0\rangle + sin(\theta_0)|1\rangle) \otimes |0\rangle \right). \quad (6)$$

$$R_1(R_0|H\rangle) = \left(I \otimes \sum_{i=0, i\neq 1}^{3} |i\rangle\langle i|) + R_y(2\theta_1) \otimes |1\rangle\langle 1| \right) \frac{1}{2}\left(|0\rangle \otimes \sum_{i=0, i\neq 0}^{3} |i\rangle\langle i| + (\cos(\theta_0)|0\rangle + sin(\theta_0)|1\rangle) \otimes |0\rangle] \right)$$

$$= \frac{1}{2}\left([|0\rangle \otimes \sum_{i=0, i\neq 0,1}^{3} |i\rangle\langle i| + (\cos(\theta_0)|0\rangle \sin(\theta_0)|1\rangle) \otimes |0\rangle + (\cos(\theta_1)|0\rangle + sin(\theta_1)|1\rangle) \otimes |1\rangle]. \right) \quad (7)$$

$$R_2(R_1R_0|H\rangle) = \left(I \otimes \sum_{i=0, i\neq 2}^{3} |i\rangle\langle i|) + R_y(2\theta_2) \otimes |2\rangle\langle 2| \right) \frac{1}{2}\left(|0\rangle \otimes \sum_{i=0, i\neq 0,1}^{3} |i\rangle\langle i| + (\cos(\theta_0)|0\rangle + sin(\theta_0)|1\rangle) \otimes |0\rangle + (\cos(\theta_1)|0\rangle + sin(\theta_1)|1\rangle) \otimes |1\rangle \right)$$

$$= \frac{1}{2}\left(|0\rangle \otimes \sum_{i=0, i\neq 0,1,2}^{3} |i\rangle\langle i| + (\cos(\theta_0)|0\rangle + sin(\theta_0)|1\rangle) \otimes |0\rangle + (\cos(\theta_1)|0\rangle + sin(\theta_1)|1\rangle) \otimes |1\rangle + (\cos(\theta_2)|0\rangle + sin(\theta_2)|1\rangle) \otimes |2\rangle \right). \quad (8)$$

$$R_3(R_2R_1R_0|0\rangle) = \left(I \otimes \sum_{i=0, i\neq 3}^{3} |i\rangle\langle i|) + R_y(2\theta_3) \otimes |3\rangle\langle 3| \right) \frac{1}{2} \left(|0\rangle \otimes \sum_{i=0, i\neq 0,1,2}^{3} |i\rangle\langle i(\cos(\theta_0)|0\rangle + sin(\theta_0)|1\rangle) \otimes |0\rangle + (\cos(\theta_1)|0\rangle + sin(\theta_1)|1\rangle) \otimes |1\rangle \quad + (cos(\theta_2)|0\rangle + sin(\theta_2)|1\rangle) \otimes |2\rangle \right)$$

$$= \frac{1}{2}|0\rangle \left(\otimes \sum_{i=0, i\neq 0,1,2,3}^{3} |i\rangle\langle i| + (\cos(\theta_0)|0\rangle + sin(\theta_0)|1\rangle \otimes |0\rangle + (\cos(\theta_1)|0\rangle + sin(\theta_1)|1\rangle \otimes |1\rangle + (cos(\theta_2)|0\rangle + sin(\theta_2)|1\rangle) \otimes |2\rangle + (cos(\theta_3)|0\rangle + sin(\theta_3)|1\rangle) \otimes |3\rangle \right). \quad (9)$$

In above equation (9), the first term enclosed in parentheses vanishes and $\sum_{i=0}^{3} |i\rangle$ are just numeric representation of two qubit states in Zeeman basis,

1. $|0\rangle = |00\rangle$
2. $|1\rangle = |01\rangle$
3. $|2\rangle = |10\rangle$
4. $|3\rangle = |11\rangle$.

Hence, the FRQI state achieved is,

$$\frac{1}{2}\left((\cos(\theta_0)|0\rangle + sin(\theta_0)|1\rangle) \otimes |00\rangle + (\cos(\theta_1)|0\rangle + sin(\theta_1)|1\rangle) \otimes |01\rangle + (\cos(\theta_2)|0\rangle + sin(\theta_2)|1\rangle) \otimes |10\rangle + (\cos(\theta_3)|0\rangle + sin(\theta_3)|1\rangle) \otimes |11\rangle \right). \quad (10)$$

Pixel value	Angle
0	0
85	$\pi/6$
170	$\pi/3$
255	$\pi/2$

TABLE I. Angles for corresponding pixels of FIG.1.

III. IMPLEMENTATION METHODOLOGY AND RESULTS

In order to begin with the implementation of FRQI, we need to have some knowledge about pixels. To put it in the simplest way possible, a pixel is the smallest unit of an image. An image can be composed of up to millions of pixels. Each pixel of a gray scale classical image is represented by a number between 0 and 255, where 0 corresponds to black and 255 corresponds to white. All the numbers in between are various shades of gray of decreasing intensities. In order to implement the FRQI circuit, lets take the example of a 2x2 classical image shown in Fig. 1. It is a gray scale classical image composed of 4 pixels. The numbers that correspond to these colors

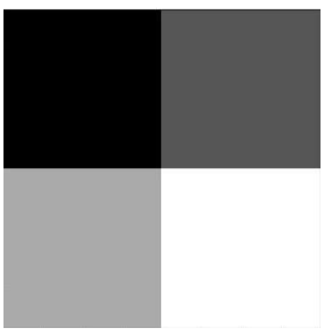

FIG. 1. A 2x2 grey scale image.

are 0, 85, 170 and 255, starting from top left, top right, bottom left and bottom right respectively. Now, the first step for the implementation of the circuit for FRQI is to convert these values into angles that will be processed by the quantum circuit. The following formula will be used to carry out this conversion:

$$\theta = \frac{numericalvalue}{255}.\frac{\pi}{2}. \tag{11}$$

Using this Equation. 12, pixel values are shown in Table. I. Substituting these values into Equation. 10 gives us the quantum state (FRQI) for the classical image displayed in Fig. 1,

$$|\psi\rangle = \frac{1}{2}|000\rangle + \frac{\sqrt{3}}{4}|001\rangle + \frac{1}{4}|101\rangle + \frac{1}{4}|010\rangle + \frac{\sqrt{3}}{4}|110\rangle + \frac{1}{2}|111\rangle. \tag{12}$$

State	$\|000\rangle$	$\|001\rangle$	$\|010\rangle$	$\|101\rangle$	$\|110\rangle$	$\|111\rangle$
Probability	0.2497	0.1919	0.06255	0.06225	0.1899	0.2437

TABLE II. Probabilities for each 3 qubit states detected.

Equation. 12 represents the quantum state in which our classical grey scale image has been encoded. This process of encoding involves passing the image through a quantum circuit shown in Figure. 2. The hadamard gates

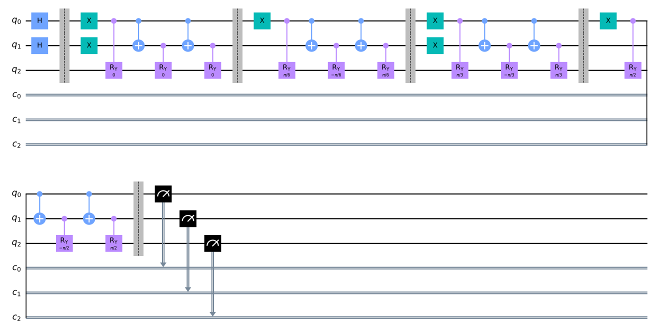

FIG. 2. Quantum circuit for a 2x2 FRQI.

creates superposition. This is followed by a combination of gates shown in right hand side of barrier in Figure. 3. In this Figure. 3 the combination of gates on the right hand side of barrier are applied in our quantum circuit and are equivalent to a single quantum gate on left hand side of barrier. This single gate is a doubled controlled Y rotation quantum gate and works for any angle.

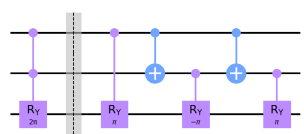

FIG. 3. A double controlled Y rotation gate can be broken down into 4 simpler gates.

A. Measurement and recovering the image

Now that classical image has been encoded, it is time to recover it back. Since we can only see what our naked eye allows us to and a quantum state is really not useful when it comes to visualizing images. Now a measurement has been made at the end of our quantum circuit in Figure. 2. This was done by running it on IBM quantum computer. The maximum number of counts allowed were used i.e. 20000. The results are shown in Figure. 4. It shows the number of times each of the 3 qubit states were detected. This is basically a probability distribution. In order to get explicit values for probabilities, each value is divided by total number of shots fired. This is shown in Table 2.

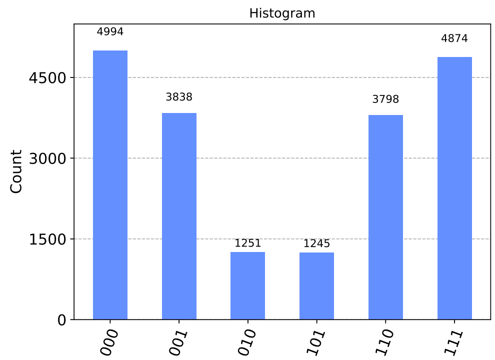

FIG. 4. States detected after measurement has been done.

$$\begin{pmatrix} 0.25 & 0.2165 & 0.125 & 0 & 0 & 0.125 & 0.2165 & 0.25 \\ 0.2165 & 0.1875 & 0.1083 & 0 & 0 & 0.1083 & 0.1875 & 0.2165 \\ 0.125 & 0.10825 & 0.0625 & 0 & 0 & 0.0625 & 0.1083 & 0.125 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.125 & 0.10825 & 0.0625 & 0 & 0 & 0.0625 & 0.10825 & 0.125 \\ 0.2165 & 0.1875 & 0.1083 & 0 & 0 & 0.1083 & 0.1875 & 0.2165 \\ 0.25 & 0.2165 & 0.125 & 0 & 0 & 0.125 & 0.2165 & 0.25 \end{pmatrix}$$

FIG. 5. Density operator corresponding to state $|\psi\rangle$.

B. Brute force approach

Now we would like to do something we call a brute force approach. This involves recovering the classical 2x2 image. We start by writing down the density operator corresponding to the quantum state in Equation. 12. This is shown in Figure. 5. Next we will reconstruct the quantum state using the Table. II. In basic principle, the modulus square of the coefficients of a quantum state gives probability and the sum is equal to 1 for normalization. We use the same principle and reconstruct the quantum state after measurement based in the probabilities and states detected.

$$\begin{aligned} |\psi^{'}\rangle = 0.4997|000\rangle + 0.4381|001\rangle + 0.2501|010\rangle \\ + 0.2495|101\rangle + 0.4358|110\rangle + 0.4937|111\rangle. \end{aligned} \tag{13}$$

Now the corresponding density operator for this state $|\psi\rangle$ after measurement is shown in Figure. 6

$$\begin{pmatrix} 0.2497 & 0.2189 & 0.1249 & 0 & 0 & 0.1247 & 0.21788 & 0.2467 \\ 0.2189 & 0.1919 & 0.1096 & 0 & 0 & 0.1093 & 0.1909 & 0.2163 \\ 0.1249 & 0.1096 & 0.0625 & 0 & 0 & 0.0624 & 0.1089 & 0.1235 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1247 & 0.1093 & 0.0624 & 0 & 0 & 0.0623 & 0.1087 & 0.1232 \\ 0.2178 & 0.1909 & 0.1089 & 0 & 0 & 0.1088 & 0.1899 & 0.2151 \\ 0.2467 & 0.2163 & 0.1235 & 0 & 0 & 0.1232 & 0.2151 & 0.2437 \end{pmatrix}$$

FIG. 6. Density operator corresponding to state$|\psi^{'}\rangle$.

State	P(j—|0⟩)	P(j—|1⟩)	pixel value(ϑ)
00	0.4997	0	0
01	0.4381	0.2495	105
10	0.2501	0.4358	150
11	0	0.4937	255

TABLE III. Pixel values calculated from the reconstructed state using Equation.]reftab:3

C. Recovering the image

Now we intend to recover the classical image back from the quantum state. For this purpose we need the pixel values to reconstruct the grey scale image. We will use the following formula,

$$\vartheta = \arccos\left(\sqrt{\frac{P(j||0\rangle)}{P(j||0\rangle) + P(j||1\rangle)}}\right).255.\frac{2}{\pi}. \tag{14}$$

The values for the pixels are calculated using Equation. 14, where $P(j||0\rangle)$ is conditional probability of observing state j given that gray value of qubit is in state $0\rangle$. Analogously, $P(j||1\rangle)$ is the conditional probability of observing j with gray value qubit in $|1\rangle$. The results of our calculations of pixel values from the reconstructed state $|\psi^{'}\rangle$ are displayed in Table. III.

D. Evaluation Metrics

Now we will use an evaluation metric to quantify how efficient this encoding process was. For this purpose we will be using fidelity. Fidelity is a measure of how close two quantum states are using their density operators. [5] Mathematically it is defined as:

$$F(\rho, \sigma) = (tr\sqrt{\rho\sigma})^2. \tag{15}$$

Our calculated value of fidelity turns out to be 0.99, which is very high. But again this was a very direct or how we would like to call it 'brute force' approach to recover the quantum state.

Using these pixel values classical gray scale value can be recovered. But we will now quantify the error in recovering this image using mean square error [6],

$$L_{MSE} = \frac{1}{n}\sum_{i=1}^{n}(x_i - y_i)^2. \tag{16}$$

In Equation. 16, x_i and y_i correspond to the pixel values for original and reconstructed image respectively. When calculated, mean square error comes out to be 200. With increase in error in pixel values, this squared loss increases quadratically.

IV. CONCLUSION

We started with a classical grey scale 2x2 image and encoded it into a quantum state called flexible representation of quantum image. This is based on the fact that a single qubit is used to encode grey scale color and 2 qubits were used to encode the position of all 4 pixels in this specific quantum image representation. A square image with 2^nx2^n size can be encoded in just $2n+1$ qubits. The preparation complexity is $O(2^{4n})$. Pixel values for the grey scale image were converted into angles to encode into the quantum state. The quantum circuit was run on IBM quantum computer and probability distribution was obtained after measurement. From this measurement we attempted to reconstruct the quantum state FRQI. The efficiency of such straight forward approach was quantified using fidelity which turned out to be 0.99. This was followed by recalculation of pixels from the reconstructed state to recover the classical image. This was quantifies through mean square error which was 200.

V. REFERENCES

[1] IBM, What is quantum computing?, `https://www.ibm.com/topics/quantum-computing` (2020).

[2] P. Q. Le, F. Dong, and K. Hirota, A flexible representation of quantum images for polynomial preparation, image compression, and processing operations, Quantum Information Processing **10**, 63 (2011).

[3] IBM, Flexible representation of quantum images (frqi) and novel enhanced quantum representation (neqr), `https://learn.qiskit.org/course/ch-applications/flexible-representation-of-quantum-images-frqi` (2020).

[4] Q. P. Le, D. Fangyang, A. Yoshinori, and H. Kaoru, Flexible representation of quantum images and its computational complexity analysis, in *25* (, 2009) pp. 185–185.

[5] Y.-C. Liang, Y.-H. Yeh, P. E. Mendonça, R. Y. Teh, M. D. Reid, and P. D. Drummond, Quantum fidelity measures for mixed states, Reports on Progress in Physics **82**, 076001 (2019).

[6] C. M. Bishop and N. M. Nasrabadi, *Pattern recognition and machine learning*, Vol. 4 (Springer, 2006).