

Project 3 Final Report – Week 4

Coupled Oscillators

In this experiment, we will investigate the oscillation of two coupled, identical masses. Springs will be used for coupling; two masses will be coupled with a spring in between them and two springs on each side to connect them to anchor points. The springs on the side will have the same spring constant, meanwhile the center spring will have a different spring constant.

Theory

For a linear oscillator, the of a mass m and a spring attached to a fix position the equation which relates to its simple harmonic motion is

$$x = A \cdot \cos(\omega_0 t + \delta), \text{ where } \omega_0 = \sqrt{\frac{k}{m}}$$

Now for a coupled oscillating system, the equations are a bit more complex.

Firstly using, $-kx = ma$, we get the following.

$$m_1 \cdot \ddot{x}_1 = -k_1 \cdot x_1 - k' \cdot (x_1 - x_2)$$

$$m_2 \cdot \ddot{x}_2 = -k_2 \cdot x_2 - k' \cdot (x_2 - x_1).$$

Where k is the spring constant of the side springs and k' is the spring constant of the center spring. m_1 and m_2 are the masses of the oscillators and x_1 and x_2 are their relative displacements from their mean position.

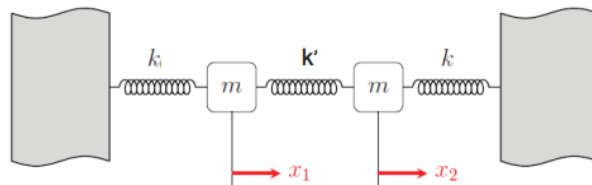


Figure 1: A coupled oscillator system

For the setup shown, we get 2 resonance frequencies for this system, ω_1 and ω_2 .

$$\omega_1 = \sqrt{\frac{k}{m}}, \omega_2 = \sqrt{\frac{k+2k'}{m}}$$

The general solution of the equation of motion is the superposition of the normal modes. Normal modes being the state of an oscillating system where both masses oscillate with the same frequency.

$$x_1 = A_1 \cos(\omega_1 t + \delta_1) + A_2 \cos(\omega_2 t + \delta_2)$$

$$x_2 = A_1 \cos(\omega_1 t + \delta_1) - A_2 \cos(\omega_2 t + \delta_2).$$

Where four constants A_1 , A_2 , δ_1 and δ_2 are set by initial conditions, outlining characteristics of Simple Harmonic Motion.

Let us consider a few special cases of oscillation.

Case 1:

Displacing the masses by the same amount and in the same direction ($x_1 = x_2$), results in no extension of the spring k' . Therefore, the oscillation is completely dependent on the extension/compression of the springs with spring constant k . Therefore, k' is not present in

$$\omega_1 = \sqrt{\frac{k}{m}}$$

Case 2:

Displacing the masses by the same amount but in the different directions ($x_1 = -x_2$), the middle spring k' is extended/compressed. Therefore, the oscillation is dependent on the extension/compression of both k and k' . Therefore, both k and k' are present in

$$\omega_2 = \sqrt{\frac{k+2k'}{m}}$$

Case 3:

Releasing one mass from rest and the other from a non-zero displacement results in an interesting phenomenon, encompassing both resonance frequencies, ω_1 and ω_2 . The displacements from the mean position of both oscillators are given by.

$$x_1 = C \cos\left(\frac{\omega_2 - \omega_1}{2}t\right) \cdot \cos\left(\frac{\omega_2 + \omega_1}{2}t\right)$$

$$x_2 = C \sin\left(\frac{\omega_2 - \omega_1}{2}t\right) \cdot \sin\left(\frac{\omega_2 + \omega_1}{2}t\right)$$

This is characteristic of Beat Phenomenon. Beats are caused by the interference between two waves of the same amplitude, travelling with the same wave speed, but having slightly different frequencies, f_1 and f_2 . In this case ω_1 and ω_2 . In this case, the energy of one oscillator is converted into the energy of the other. This results in the Oscillators moving with a common “beat frequency”.

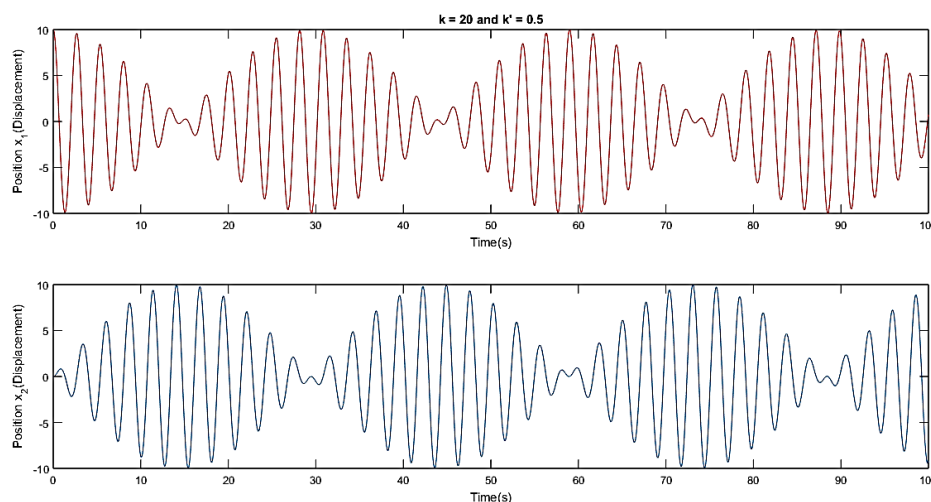


Figure 2: Sample MATLAB plots that show the Beat Phenomenon

This MATLAB plot with arbitrary values of m , C , k and k' , helps us better visualize this phenomenon. The maximum of the displacements of the graph is equal to the magnitude of amplitude C . The sum of the displacements of both graphs at a point will always be equal to C . This can be seen in the equations where the displacement of one graph is dependent on Cosine and the other is dependent on Sine.

Experimental Analysis

To test our Theory, let's examine the oscillation of two coupled gliders on a frictionless surface, in this case an air rail.

Oscillator Design

We designed two identical 3D-printed masses (oscillators) that were separately glued atop 2 freely moving platforms. For this purpose, we used Fusion 360 to design 2 laser cut slabs with dimensions 12cm x 5.55 cm and 12cm x 6cm. with width 0.45cm. These will be glued together at their ends perpendicular to each other to form a right angle in between them. These will act as our gliders.

We used fusion 360 to design our masses as well. The following are the resulting models after taking into account the center of mass of the oscillator and the gliding platform which must be in line with the springs and the anchor point.

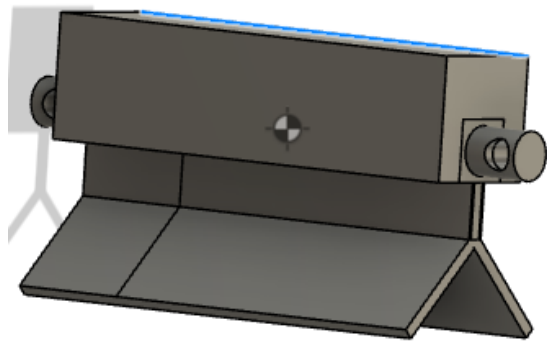


Image 1: 3D Model in Fusion 360

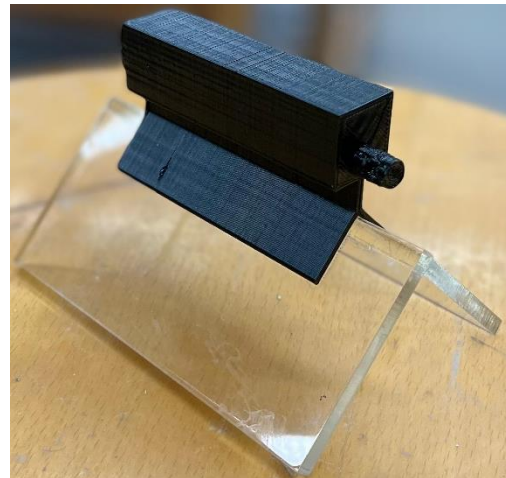


Image 2: Final mass-glider system

The measured masses of our mass-glider system were 98.8 and 99.0 g. (approximately similar) (we will use the average of 98.9 g in all later calculations)

Spring Design:

Considerations:

- Springs had to be of a sufficient length so they can stretch comfortably between their hooking points without deforming.
- The springs should not be too long and should not touch the Air rail at any point.
- The spring constant should be sufficient enough to allow for the oscillations of the masses.
- For this experiment, we have assumed the 2 springs connected to the anchor points to have the same spring constant, and they will be designed as such.
- The middle spring will have a different k value (k')

Designing the springs and calculating the spring constant:

The unstretched lengths for the side springs (5.0 cm) were chosen to be the same so that their spring constant is approximately similar. (1.75 N/m and 1.77 N/m) (we will use the Average value $k = 1.76 \text{ N/m}$)

The spring constant for the center spring has been chosen to be different than the side springs. Since it was made from the same material, its length (8.0cm) was increased to decrease the spring constant ($k' = 1.04 \text{ N/m}$)

Spring constant was calculated using Hooke's law ($F = kx$) by hooking the spring onto a stand, attaching known, measured, varying masses to its end, measuring the new lengths, comparing it with the original lengths and finding the extension. The data could then be imported into any plotting software, with plots of F (convert mass to Newtons) against x (extension). The gradient would then just be

$$m = \frac{F}{x} = k$$

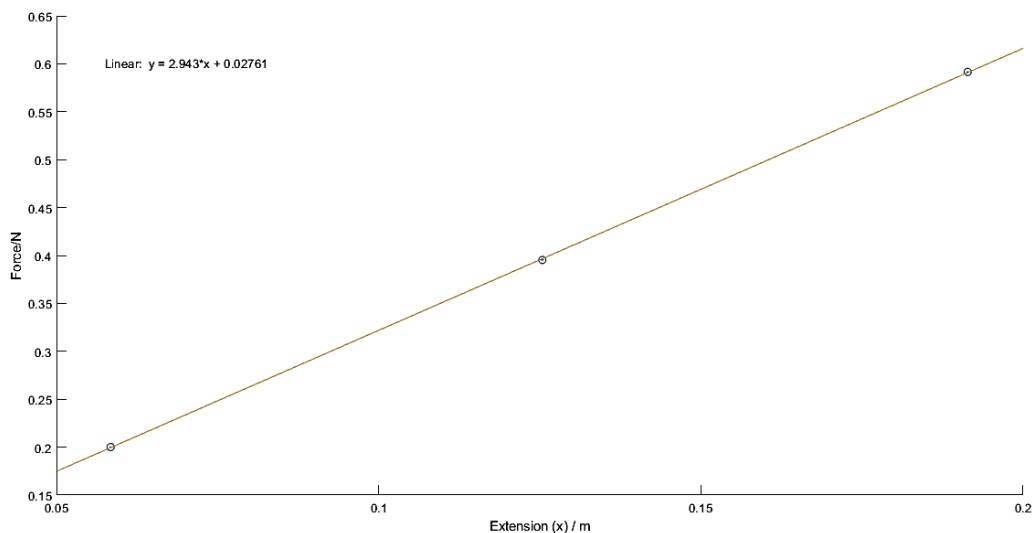


Figure 3: Sample plot in MATLAB which shows how data points may be plotted and gradient (k) may be found using a line of best fit

Final Experimental Design and Calculations:

We mounted the masses and springs in their respective positions on the Air Rail.

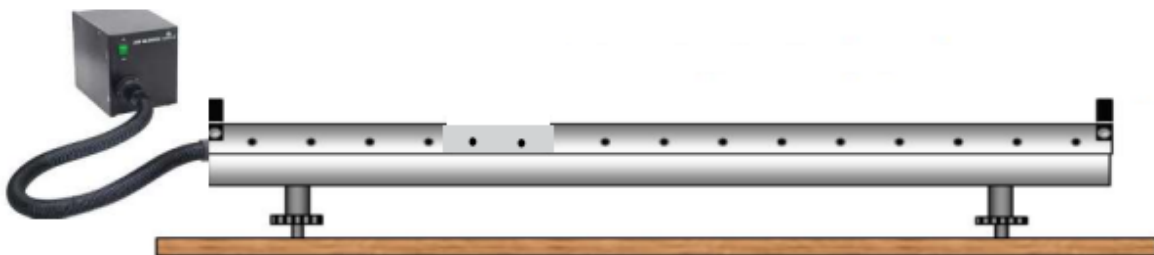


Image 3: The Air rail powered by a blower which makes the rail nearly frictionless

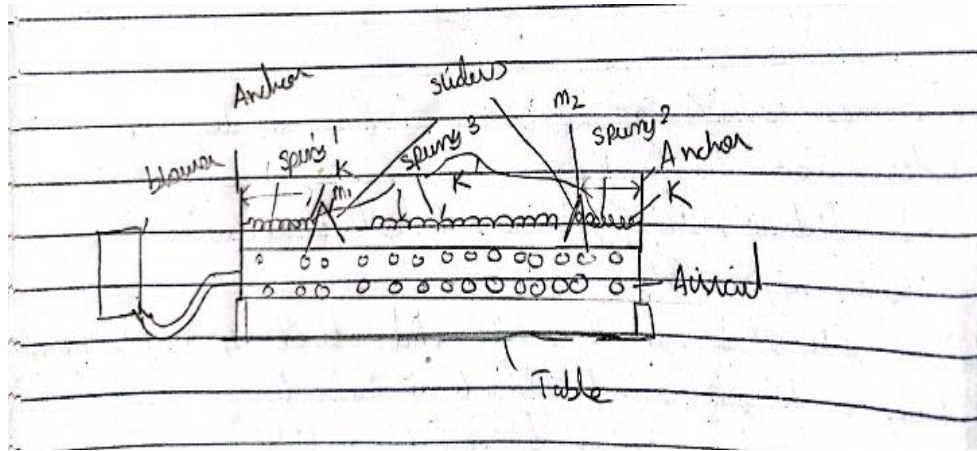


Image 4: Rough schematic of the design of the assembled experiment apparatus



Image 5: The final assembled apparatus

Data collection and Final calculations

In order to examine each case of oscillation which we outlined previously; we used the method of timing the oscillations for a fixed number of oscillations.

Before taking each measurement for time, we turn on the blower and ensure the masses are at their mean position and at rest. Then we displace the masses manually based on the Case being investigated. We used meter rules to ensure the magnitude of the displacement is the same. The same person holds and lets go of the oscillators. A different person mans the stopwatch throughout. We take the time for 10 oscillations and find the time period T by $t_{\text{avg}}/10$. Repeat the experiment 3 times for each case and find the average T .

Case 1

We displace both oscillators by the same distance in the same direction.

$$t_1 = 15.7 \pm 0.4 \text{ s}, t_2 = 15.8 \pm 0.4 \text{ s}, t_3 = 15.7 \pm 0.4 \text{ s}, t_{\text{avg}} = 15.7 \pm 0.4 \text{ s}, T = 1.57 \pm 0.05 \text{ s}$$

$$\omega_1 = 2\pi / T$$

We find the experimental $\omega_1 = 4.0 \pm 0.1 \text{ s}^{-1}$

The theoretical value $\omega_1 = 4.2 \text{ s}^{-1}$

Case 2

We displace both oscillators by the same distance in different directions.

$$t_1 = 10.4 \pm 0.40 \text{ s}, t_2 = 10.4 \pm 0.40 \text{ s}, t_3 = 10.4 \pm 0.40 \text{ s}, t_{\text{avg}} = 10.4 \pm 0.40 \text{ s}, T = 1.04 \pm 0.04 \text{ s}$$

$$\omega_2 = 2\pi / T$$

We find the experimental $\omega_2 = 6.0 \pm 0.2 \text{ s}^{-1}$

The theoretical value $\omega_2 = 6.2 \text{ s}^{-1}$

Case 3

We displace one oscillator while the other remains at the rest mean position.

$t_1 = 29.5 \pm 0.4 \text{ s}$, $t_2 = 29.7 \pm 0.4 \text{ s}$, $t_3 = 29.6 \pm 0.4 \text{ s}$, $t_{\text{avg}} = 29.6 \pm 0.4 \text{ s}$, $T = 2.96 \pm 0.04 \text{ s}$

$$\underline{\Omega} = 2\pi / T$$

We find the experimental $\underline{\Omega} = 2.12 \pm 0.03 \text{ s}^{-1}$

The theoretical value $\underline{\Omega} = \omega_2 - \omega_1 = 2.02 \text{ s}^{-1}$

Reasons for experimental and theoretical variations:

- Slight deformation of springs when mounting them onto masses.
- Inaccuracies in measuring the displacements in Case 1 and 2.
- The track is not completely frictionless.
- The gliders do not have completely smooth surfaces
- Air rail has ridges at connection points which may hinder motion
- Difficulty in measuring T for Case 3.
- Air resistance causes damping.

Method to more accurately measure trajectories of the coupled oscillators

Instead of timing the oscillations, we may measure the values of the displacements of each oscillator. Doing this manually is difficult, therefore we use displacement sensors (PhysDisp). We mounted the sensors at fixed positions near the left and right anchor points.

We used sheets of paper to act as our position markers, attaching them to our oscillating masses. The configuration was setup as shown

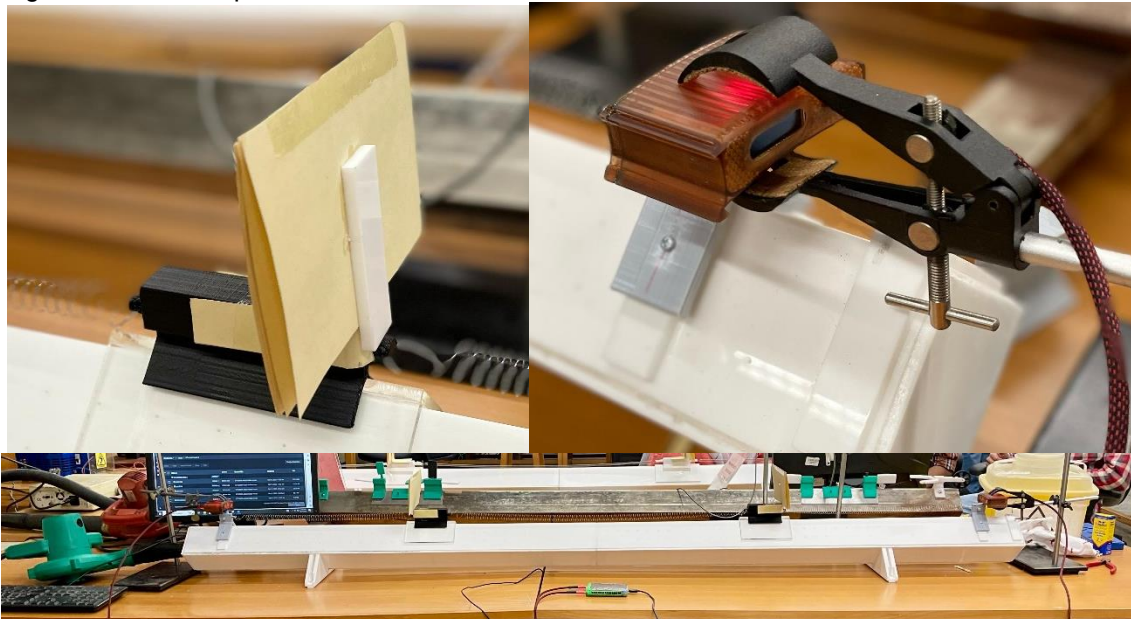


Image 6: Modified Setup used to measure Displacements

The following are the plots made by the PhysLogger software using input from the PhysDisp.

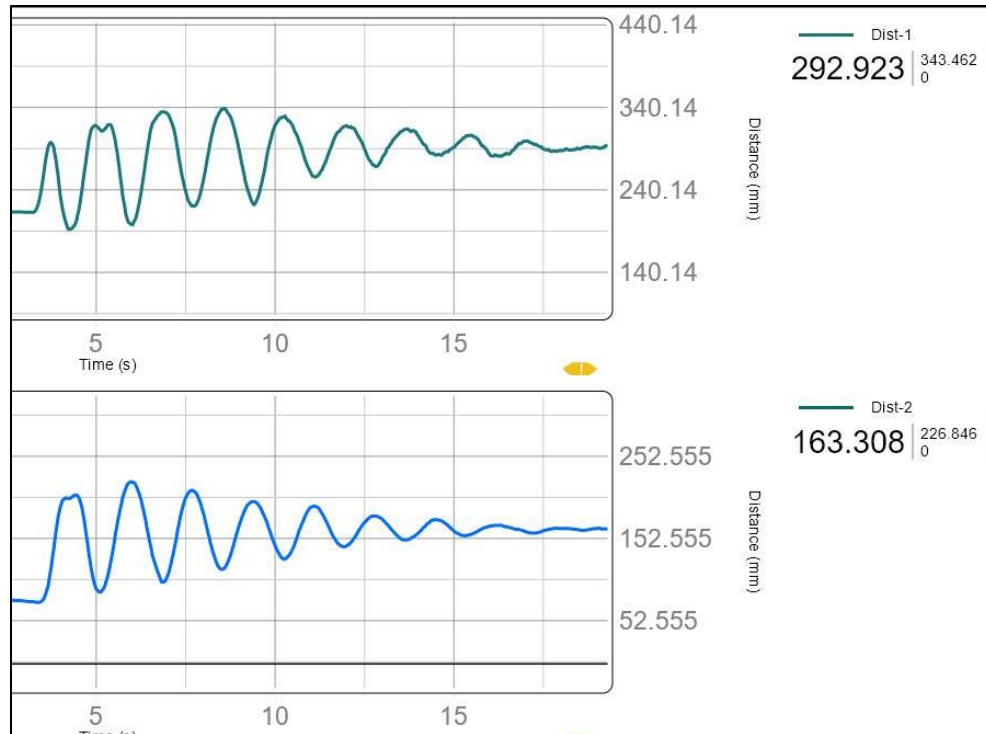


Figure 4: Graph for Case 1

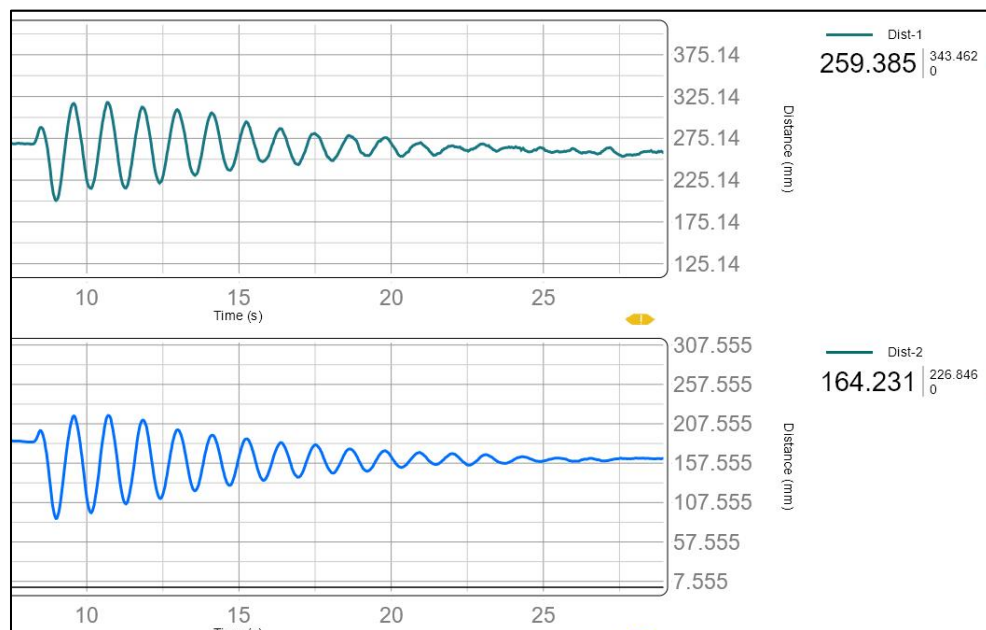


Figure 5: Graph for Case 2

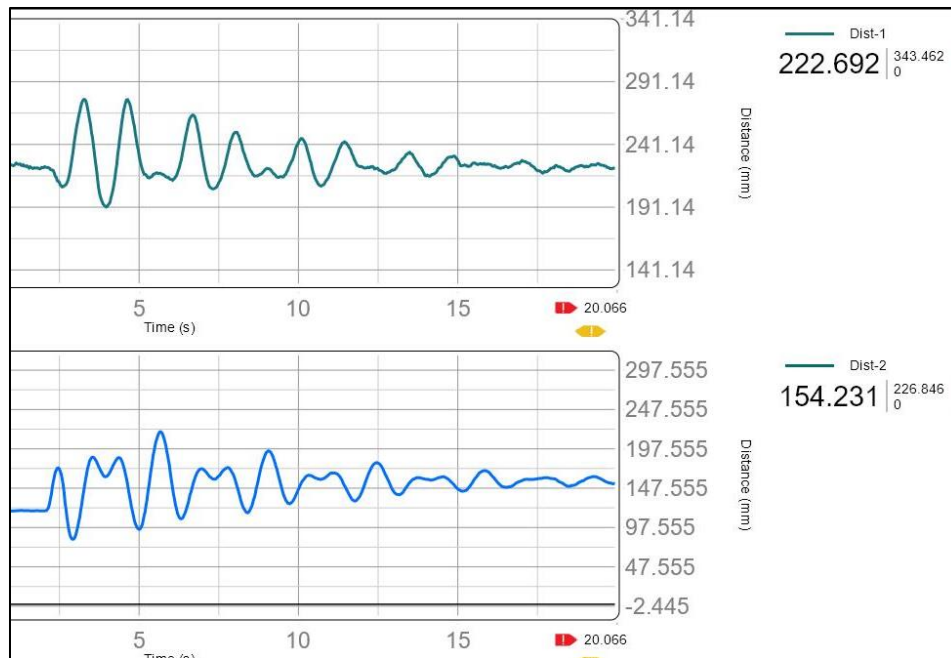


Figure 6: Graph for Case 3

The oscillations slow down and the oscillators eventually come to a stop. This is due to the heightened effect of the damping caused by Air resistance because of the attached sheets of paper and their relatively large surface area.

Let's use a simulation in Web Vpython to see what the effect would be like without Air resistance/Damping.

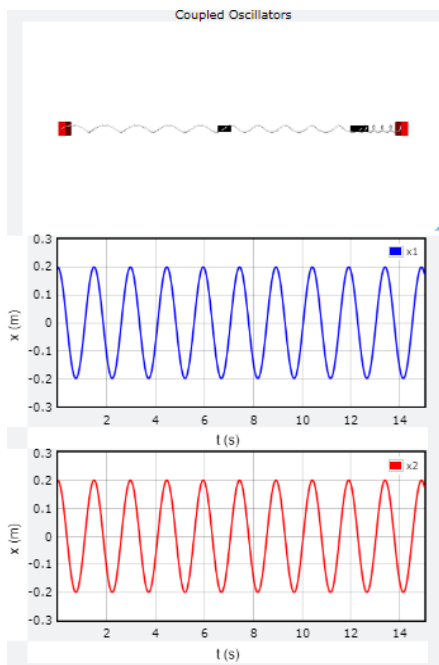


Figure 7: Simulation for Case 1

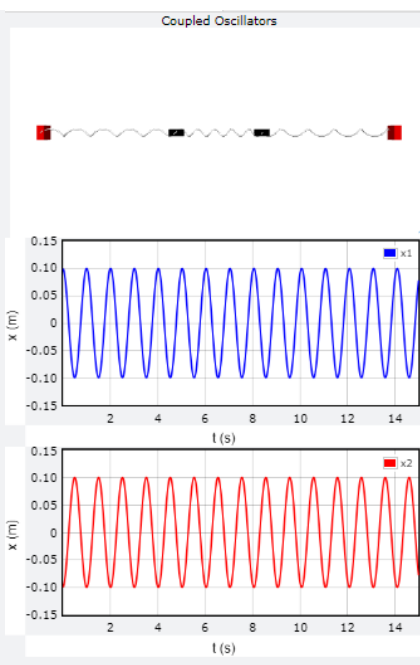


Figure 8: Simulation for Case 2

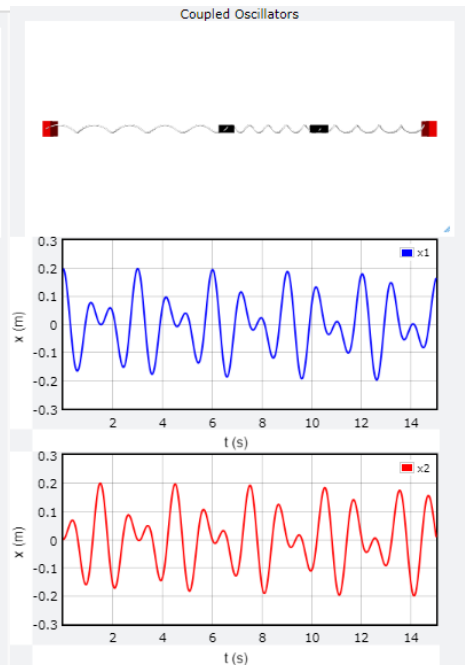


Figure 9: Simulation for Case 3


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1 Web VPython 3.2
2 scene = canvas(title='Coupled Oscillators',
3 | width=500, height=250, background=color.white)
4 g1 = graph(xtitle="t (s)",ytitle="x (m)",width=500, height=250)
5 f1 = gcurve(color=color.blue, label="x1")
6 g2 = graph(xtitle="t (s)",ytitle="x (m)",width=500, height=250)
7 f2 = gcurve(color=color.red, label="x2")
8
9 k = 1.76
10 m = 0.0989
11 k_dash = 1.04
12 left_wall = box(pos=vector(-.5,0,0),size=vector(0.02,0.04,0.02), color=color.red)
13 right_wall = box(pos=vector(0.5,0,0),size=vector(0.02,0.04,0.02), color=color.red)
14 x1 = 0.2
15 x2 = 0
16 x1_v = 0
17 x2_v = 0
18 mass1 = box(pos=vector(-0.2+x1,0,0),size=vector(0.04,0.02,0.02), color=color.black)
19 mass2 = box(pos=vector(0.2+x2,0,0),size=vector(0.04,0.02,0.02), color=color.black)
20 spring1=helix(pos=left_wall.pos, axis=mass1.pos-left_wall.pos, radius=0.01, thickness=0.003)
21 spring2 = helix(pos=mass1.pos, axis=mass2.pos-mass1.pos, radius=0.01,thickness=0.003)
22 spring3 = helix(pos=right_wall.pos, axis=mass2.pos-right_wall.pos, radius = 0.01, thickness=0.003)
23
24 t = 0
25 dt = 0.01
26 while t<15:
27     rate(100)
28
29     F1 = -k*x1-k_dash*(x1-x2)
30     F2 = -k*x2 - k_dash*(x2-x1)
31
32     x1_acc = F1/m
33     x2_acc = F2/m
34
35     x1_v = x1_v + x1_acc*dt
36     x2_v = x2_v + x2_acc*dt
37     x1 = x1 + x1_v*dt
38     x2 = x2 + x2_v*dt
39     mass1.pos = vector(-0.2+x1,0,0)
40     mass2.pos = vector(0.2+x2,0,0)
41
42     spring1.axis=mass1.pos-left_wall.pos
43     spring2.pos=mass1.pos
44     spring2.axis=mass2.pos-mass1.pos
45     spring3.axis = mass2.pos - right_wall.pos
46
47     t = t + dt

```

Image 7: Vpython code used for the simulation

Contributions:

Abdul Nafae Imran: 3D design and Collection of Data

Abdul Moez Khurshid: Setup of Apparatus and Collection of Data

Muneeb Ul Haq: Uncertainties and Measurements

Zaryab Gohar: Reports, finding spring constants and simulation