Question 1

For a light beam of wavelength 630 nm and power 23 mW, how many photons are emitted per second?

Question 2

Given that $z = 3 + 5i$ and $y = 5 + 8i$,

(a) find the product of z and y.

(b) find the conjugates of z and y, z^* and y^* respectively.

(c) find the sum of each number with its conjugate; $z + z^*$ and $y + y^*$. Are your answers always real or always imaginary?

(d) find the difference between each number and its conjugate; $z - z^*$, and $y - y^*$. Are your answers always real or always imaginary?

(e) find the products of z and y with their respective conjugates; zz^* and $yy*$.

(f) find z/y and y/z , give your answer in the form $a + ib$.

(g) find the magnitudes of the numbers, $|y|$ and $|z|$. Compare your answers with part (e).

Question 3

Convert the following to $Re^{i\theta}$ form.

(a) $3 + 4i$

(b) $5 + 9i$

(c) $5 - 9i$ Compare your result with part (b).

(d) $-9 + 2i$

(e) $9 - 2i$ Compare with part (d).

(f) $-5 - 3i$

(g) $5 - 3i$ (you should be able to guess this now!)

(h) $1/i$

Question 4

Convert the following to $a + ib$ form.

(a) $5e^{i\pi/6}$

(b) $9e^{i\pi}$

(c) $2e^{-i\pi/4}$

Question 5

Given that

$$(Re^{i\theta})(Se^{i\phi}) = RSe^{i(\phi+\theta)}.$$

Draw the following calculations on an argand diagram with a real and imaginary axis. First, draw both the numbers and then their products.

(a)
$$(5e^{i\pi/5})(e^{i\pi/2})$$

(b)
$$(e^{i\pi/4})(e^{i\pi/6})$$

(c) What happens to a complex number upon multiplication with $e^{i\theta}$? Describe physically.

(d) Prove that
$$|(ze^{i\theta})|^2 = |z|^2$$

Question 6

Let us implement our newly learned quantum mechanical skills on the Mach-Zehnder interferometer shown in Figure 1. In class, we defined the operation of a beam-splitter on two basis states of a photon. The upward and downward propagating photon states were defined as $|0\rangle$ and $|1\rangle$ respectively. Suppose that the action of the beam-splitter on the two states is defined as:

$$|0\rangle \mapsto \frac{1}{\sqrt{2}}\big(\,|0\rangle + |1\rangle\,\big)$$
$$|1\rangle \mapsto \frac{1}{\sqrt{2}}\big(\,|0\rangle - |1\rangle\,\big).$$

Similarly, a mirror inter-converts:

$$|0\rangle \mapsto |1\rangle$$
$$|1\rangle \mapsto |0\rangle\,.$$

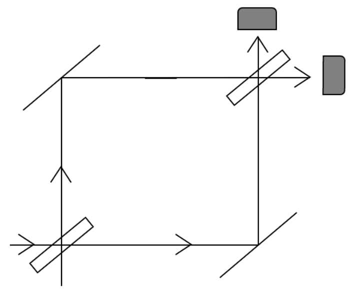

Figure 1: Schematic diagram for the Mach-Zehnder interferometer.

(a) Find the outcome probability at both the detectors given that the input state is an upward propagating state, i.e, $|1\rangle$.

(b) What if the input state is a superposition state given by $\frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big)$?

(c) Now, let us block one of the path at the location shown in Figure 2. Given that the role of a blocker is to absorb a photon, re-evaluate the previous parts using this information.

(d) Do the probabilities add up to one? If not, please explain why.

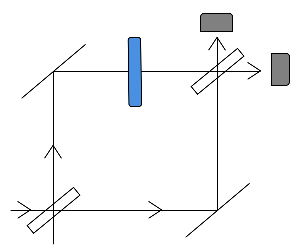

Figure 2: Schematic diagram for the Mach-Zehnder interferometer with a blockage.

Question 7

Now let's consider a slight modification of the system defined in the previous question. The operation of the BS on the basis states is now defined as:

$$|0\rangle \mapsto \frac{1}{\sqrt{2}}\big(|0\rangle + i\,|1\rangle\big)$$
$$|1\rangle \mapsto \frac{1}{\sqrt{2}}\big(i\,|0\rangle + |1\rangle\big).$$

(a) Determine the output of the state when the incoming photon is in the state $|0\rangle$.

Question 8

It may have come as a surprise that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

for $\theta \in \mathbb{R}$. Let us try to motivate this result.

(a) One way to define e^x for $x \in \mathbb{R}$ is by the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

We will also take this to be our definition for $x \in \mathbb{C}$. Write out the first few terms of the expansion for $e^{i\theta}$.

(b) Write out the first few terms of the Maclaurin series (if you haven't seen these before, look it up!) for $\cos(\theta)$ and $\sin(\theta)$.

(c) Use the previous part to write out $\cos(\theta) + i\sin(\theta)$.

(d) Compare the result of the previous part with part (a). Do your results match? Can you show that they will match if you keep on writing out the expansions?

(e) Food for thought: Is the definition in part (a) the only definition of e^x for $x \in \mathbb{C}$? Moreover, does the definition given ever run into trouble? As in, it is an infinite sum. How are we sure that it always adds up to something finite? These questions fall much beyond the scope of this course, but you are welcome to explore them.

Question 9

Consider a beam of spin-$\frac{1}{2}$ particles with mass m and horizontal velocity, $v_0\hat{\boldsymbol{y}}$, heading into a variable magnetic field pointing in the z-direction, $|\boldsymbol{B}(z)| = gz$. The axes are as shown in the diagram with $z = 0$ at the bottom of the upper half of the magnet. All values quoted are in their SI units.

(a) What are the possible values of potential energy for one such particle in this magnetic field? Answer in terms of g and μ_s. μ_s is the spin magnetic moment of the particle in the z-direction and is given by kS, where k is some constant, and S is the spin angular momentum in the z-direction.

(b) Calculate the possible forces one such particle can experience due to the magnetic field.

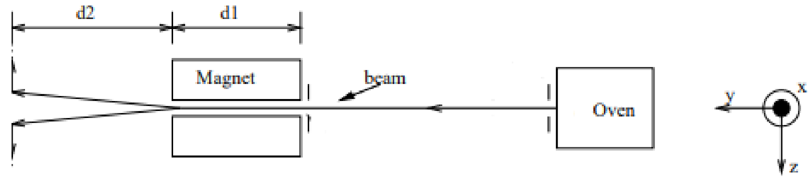

Figure 3: Stern Gerlach Sketch

(c) By looking at part (b), why do you think the Stern Gerlach setup needs a variable magnetic field?

(d) You now have the possible forces one such particle can experience, its mass, and its initial velocity. Given that the particle interacts with the magnetic field while it crosses distance $d1$, find the possible angles of deflection for these particles. Let the angle of deflection be zero if the particle continues in its horizontal path. Let deflections upwards be represented by positive angles, and deflections downwards be represented by negative angles.

(e) If we replaced our spin-$\frac{1}{2}$ particles with spin-1 particles (particles with three possible spin magnetic moments), how many different angles of deflection will be found?

Question 10

Suppose we have a system described by a four-dimensional quantum space. All we need to understand from this statement is that the state of our system will be described by four-dimensional vectors (for example, vectors in the xy-plane are two-dimensional).

(a) Suppose the vectors $|0\rangle, |1\rangle, |2\rangle, |3\rangle$ form a basis for our quantum space. What does it mean to be a basis?

(b) Suppose that each of these basis vectors is normalized. What does this tell you about $\langle i|i\rangle$ for $i = 0, 1, 2, 3$?

(c) Suppose also that these vectors are orthogonal to each other. What does this tell you about $\langle i|j\rangle$ when $i \neq j$?

(d) You are given a state

$$|\psi\rangle = \frac{1}{\sqrt{10}}|0\rangle + \frac{3}{10}|1\rangle + \frac{1}{\sqrt{5}}|2\rangle + k|3\rangle .$$

To be a legitimate quantum state, $|\psi\rangle$ should be normalized. Find k.

(e) Suppose we do a measurement on our system which is described by $|\psi\rangle$. What is the probability that we measure the system to be in the state $|0\rangle$? $|3\rangle$?