PHY104 - Midterm Exam

Each question carries the same mark. There are 13 questions.

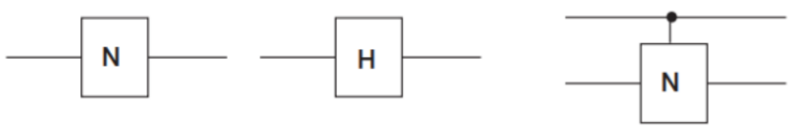
Fill in the multiple-choice answer sheet with a pencil.

You can only pick one option per question

Your Variant Code: 10. Write this on your multiple-choice answer sheet.

For this exam, we use the following conventions.

$$|D\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|A\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$
$$|L\rangle = \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}$$
$$|R\rangle = \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}}$$



Input	Output
$|0\rangle$	$|1\rangle$
$|1\rangle$	$|0\rangle$

Input	Output
$|0\rangle$	$\frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right)$
$|1\rangle$	$\frac{1}{\sqrt{2}}\left(|1\rangle - |0\rangle\right)$

Input	Output
$|0\rangle|0\rangle$	$|0\rangle|0\rangle$
$|0\rangle|1\rangle$	$|0\rangle|1\rangle$
$|1\rangle|0\rangle$	$|1\rangle|1\rangle$
$|1\rangle|1\rangle$	$|1\rangle|0\rangle$

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}.$$

$$|\psi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

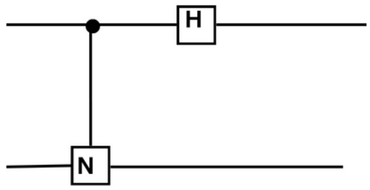
For three qubit states, the state $a\,|000\rangle + b\,|001\rangle + c\,|010\rangle + d\,|011\rangle + e\,|100\rangle + f\,|101\rangle + g\,|110\rangle + h\,|111\rangle$ **can be written as a column vector with entries in the same order as they are written out.**

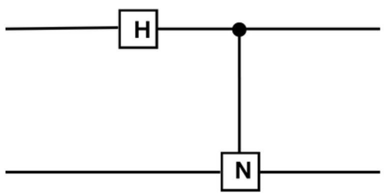
---

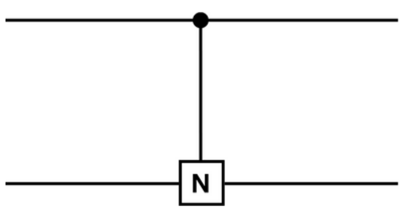
1. Which of the following circuits converts an entangled state

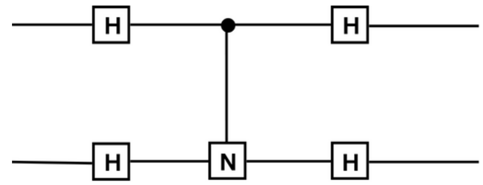
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

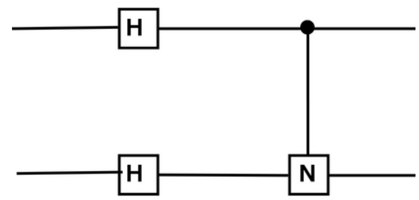
to $|00\rangle$?

A. 

B. 

C. 

D. 

E. 

---

2. In which of the following states are the first and second qubit entangled?

A. $\frac{1}{\sqrt{2}}\Big(|011\rangle + |000\rangle \Big)$

B. $\frac{1}{\sqrt{2}}\Big(|010\rangle + |011\rangle \Big)$

C. $\frac{1}{\sqrt{2}}\Big(|000\rangle + |101\rangle \Big)$

D. $\frac{1}{\sqrt{2}}\Big(|001\rangle + |010\rangle \Big)$

E. None of the options are correct

---

3. If $|\psi\rangle$ is a normalized state given by

$$|\psi\rangle = a\,|010\rangle + \frac{1}{2}\,|011\rangle - \frac{1}{2\sqrt{2}}\,|100\rangle - \frac{1}{2\sqrt{2}}\,|101\rangle\,.$$

The value of a could be: A. 0 B. $\frac{e^{i\phi}}{2}$ C. $\frac{e^{i\phi}}{\sqrt{2}}$ D. $\frac{e^{-i\phi}}{2}$ E. $\frac{1}{2}$

---

4. We have a superposition of the two Bell states $|\psi\rangle = \frac{|\Phi^+\rangle + |\Phi^-\rangle}{\sqrt{2}}$, where

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|HH\rangle + |VV\rangle\Big)$$

and

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\Big(|HH\rangle - |VV\rangle\Big).$$

Identify the correct statement from the following:

A. Such a state $|\psi\rangle$ is physically unrealizable.

B. $|\psi\rangle$ is a separable (non-entangled) state.

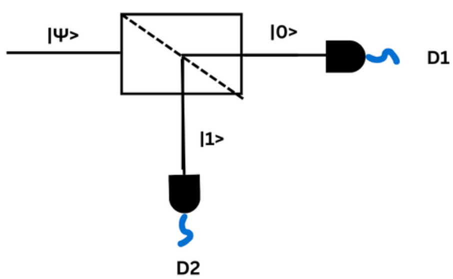
C. $|\psi\rangle$ is a fully entangled state.

D. $|\psi\rangle$ is a partially entangled state.

E. None of the statements are correct.

---

5. I have an input quantum state $|\psi\rangle$ that is either $|0\rangle$ or $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$. The state $|\psi\rangle$ is fed into a 50:50 beam splitter that has two output ports $|0\rangle$ and $|1\rangle$ as shown in the figure. Which of the following statements is **true**?



A. If D_1 clicks, the state will most definitely be $|0\rangle$.

B. If D_1 clicks, the state will most definitely be $\frac{|0\rangle+|1\rangle}{2}$.

C. If D_2 clicks, the state will most definitely be $\frac{|0\rangle+|1\rangle}{2}$.

D. If D_2 clicks, the state may still be $|0\rangle$, with some small probability.

E. If D_2 clicks, the state will most definitely be $|0\rangle$.

---

6. A state $|\psi\rangle = a|0\rangle + b|1\rangle$ (where $a \neq b, a \neq 0, b \neq 0$) is measured in the $\{|0\rangle, |1\rangle\}$ basis. This means that it passes through analyzing device that has $|0\rangle$ and $|1\rangle$ channels and detectors connected to them as shown in the figure. The detectors D_1 and D_2 have probabilities P_1 and P_2, respectively, to click.

   Another experiment is performed in which $|\psi\rangle$ is replaced by another input state, which now yields a **different** value of P_2.

   Which of the following could be this new state?

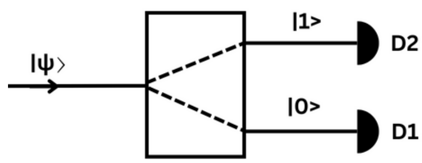
   A. $-a|0\rangle - b|1\rangle$

   B. $a|0\rangle + e^{i\phi}b|1\rangle$, $\phi \neq 0, \pi$

   C. $-a|0\rangle + b|1\rangle$

   D. $a|0\rangle - b|1\rangle$

   E. $b|0\rangle + a|1\rangle$



---

7. Which of the following states is orthogonal to $\cos(\beta)|0\rangle + e^{i\phi}\sin(\beta)|1\rangle$?

   A. $-\sin(\beta)|0\rangle + e^{i\phi}\cos(\beta)|1\rangle$

   B. $\cos(\beta)|0\rangle + e^{-i\phi}\sin(\beta)|1\rangle$

   C. $\sin(\beta)|0\rangle + e^{-i\phi}\cos(\beta)|1\rangle$

   D. $\cos(\beta)|0\rangle - e^{-i\phi}\sin(\beta)|1\rangle$

   E. $-\cos(\beta)|0\rangle + e^{i\phi}\sin(\beta)|1\rangle$

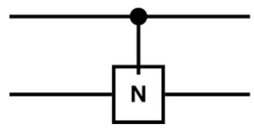
---

8. A phase gate $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ acts on $|\mathrm{L}\rangle$. The output state is

   A. $|\mathrm{V}\rangle$ B. $|\mathrm{D}\rangle$ C. $|\mathrm{A}\rangle$ D. $|\mathrm{R}\rangle$ E. $|\mathrm{H}\rangle$

---

9. The diagram shows a controlled-NOT gate. The first qubit in the tensor product is on the top quantum wire, and the second is on the bottom wire. The gate is represented by which of the following matrices?



A. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

B. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

D. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

E. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

---

10. Which of these sets do not form a basis for a qubit's quantum space?

A. $|D\rangle, |L\rangle$

B. $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle, -\sin(\theta)|0\rangle + \cos(\theta)|1\rangle$

C. $|R\rangle, |L\rangle$

D. $|0\rangle, |1\rangle$

E. $|D\rangle, |A\rangle$

---

11. A pair of photons is produced in the entangled state $\frac{|HH\rangle+|VV\rangle}{\sqrt{2}}$. One photon goes to Bob. He measures along the $|H\rangle$ or $|V\rangle$ axis. Which of the following statements for Bob is true?

   A. $P(|H\rangle) = 1$ and $P(|V\rangle) = 0$.

   B. $P(|H\rangle) = 1/2$ and $P(|V\rangle) = 1/2$.

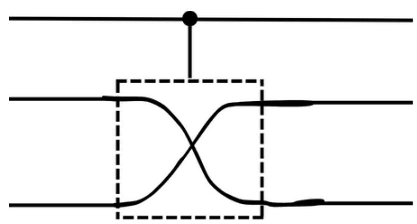
   C. $P(|H\rangle) = 0$ and $P(|V\rangle) = 1$.

   D. $P(|H\rangle) = 1/4$ and $P(|V\rangle) = 3/4$.

   E. $P(|H\rangle) = 3$ and $P(|V\rangle) = 0$.

---

12. A Fredkin gate swaps the second and third qubits if the first qubit is in the state $|1\rangle$. It is diagrammatically shown in the figure. Its unitary matrix can be written as:



A. $\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix}$

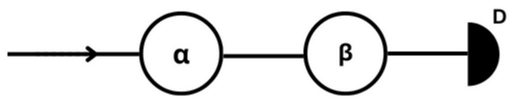
B. $\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$

C. $\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0 \end{pmatrix}$

D. $\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix}$

---

13. A photon in the polarizer state $|\varphi\rangle = |H\rangle$ sees two polarizers whose optic axes are at α and β, respectively, to the horizontal. The polarizers create a new quantum state whose polarization is parallel to the optic axes of polarizer. Detector D detects the presence of photons (in any polarisation). What is the probability that D clicks?



A. $cos^2(\alpha)cos^2(\beta)$ B. $cos(\alpha)cos(\beta)$ C. 0 D. $cos^2(\alpha)cos^2(\beta{+}\alpha)$ E. $cos^2(\alpha)cos^2(\beta{-}\alpha)$

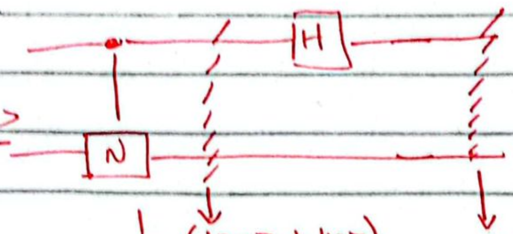
PHY 104

Variant 10

(Variant 11, 12, 13 have the same questions with the order of questions & answers permuted).

1) A

$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$



$\frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$ $\quad$ $|0\rangle|0\rangle$

$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle$

The easiest way to see this is ~~that~~ to look for the circuit that reverses $|00\rangle \longrightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$.
This was discussed in the Problem set.

2) E. None of the above.

$A \to \cancel{\frac{1}{\sqrt{2}}}\, |0\rangle \otimes \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$B \to |0\rangle \otimes |1\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$C \to \frac{1}{\sqrt{2}}\left(|0_A 0_B 0_C\rangle + |1_A 0_B 1_C\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|0_B\rangle \otimes |0_A 0_C\rangle + |0_B\rangle |1_A\rangle |1_C\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|0_B\rangle \otimes \left(|0_A 0_C\rangle + |1_A 1_C\rangle\right)\right).$

$D \to \frac{1}{\sqrt{2}}\left(|0\rangle \otimes \left(|01\rangle + |10\rangle\right)\right).$

3. $|a|^2 + \left|\frac{1}{2}\right|^2 + \left|\frac{-1}{2\sqrt{2}}\right|^2 + \left|\frac{-1}{2\sqrt{2}}\right|^2 = 1$

$|a|^2 + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$

$|a|^2 + \frac{1}{2} = 1$

$\boxed{|a|^2 = \frac{1}{2}}$ C. works. $C = \frac{e^{i\phi}}{\sqrt{2}}$

since $\left|\frac{e^{i\phi}}{\sqrt{2}}\right|^2 = \frac{e^{i\phi}}{\sqrt{2}} \cdot \frac{e^{-i\phi}}{\sqrt{2}}$

$= \frac{1}{2}$

4. B. $|\psi\rangle$ is seperable.

since $|\psi\rangle = \frac{|\phi^+\rangle + |\phi^-\rangle}{\sqrt{2}}$

$= \frac{1}{(\sqrt{2})^2}\left(|HH\rangle + \cancel{|VV\rangle} + |HH\rangle - \cancel{|VV\rangle} \right)$

$= \frac{2}{2}\left(|HH\rangle \right)$

$= |HH\rangle.$

5. if input $|0\rangle$:

$$|0\rangle \xrightarrow{B.S} \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle).$$

D_1 clicks with Prob $= \frac{1}{2}$

$P(D_2) = \frac{1}{2}$.

if input $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$:

$$\frac{|0\rangle+|1\rangle}{\sqrt{2}} \xrightarrow{B.S} |0\rangle$$

$P(D_1) = 1$, $P(D_2) = 0$

D_2 only clicks if input $= |0\rangle$ so $\underline{\underline{E}}$.

6. $|\psi\rangle = a|0\rangle + b|1\rangle$.

D_2 ~~P_0~~ clicks if $|1\rangle$.

so $P(D_2)$ with given state $= |b|^2$.

Option E $= b|0\rangle + a|1\rangle$ works.

Since $P(D_2)$ then is $|a|^2$.

7. Option A works.

Since

$$\left(-\sin\beta\langle 0| + e^{-i\phi}\cos\beta\langle 1|\right)\left(\cos\beta|0\rangle + e^{i\phi}\sin(\beta)|1\rangle\right)$$

$$= -\sin\beta\cos\beta + \cos\beta\sin\beta = 0.$$

CamScanner

8. Using the matrix, we see that

$$|0\rangle \xrightarrow{\text{Phase gate}} |0\rangle$$

$$|1\rangle \longrightarrow i|1\rangle$$

So $|L\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$

$$\xrightarrow{\text{Phase gate}} \frac{1}{\sqrt{2}}\left(|0\rangle + i\left(i|1\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

since $i^2 = -1$ (by def.)

So $|A\rangle$ is the output. <u>option C</u>.

9. $|00\rangle \xrightarrow{CNOT_{12}} |00\rangle$

$|01\rangle \longrightarrow |01\rangle$

$|10\rangle \longrightarrow |11\rangle$

$|11\rangle \longrightarrow |10\rangle$.

in

$$\begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline |00\rangle & & & & \\ |01\rangle & & & & \\ |10\rangle & & & & \\ |11\rangle & & & & \end{array} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

out

So <u>option C</u>.

10. $|D\rangle, |L\rangle$ do <u>not</u> form an <u>orthonormal</u> <u>basis</u>.

$$|D\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad ; \quad |L\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$\langle D|L\rangle = \frac{1}{2}(1+i) \neq 0.$$

11. In $\frac{|HH\rangle + |VV\rangle}{\sqrt{2}}$, it does <u>not</u> matter which photon goes to Bob. So let's take the <u>first qubit</u> goes to Bob. Then, Bob measuring $|H\rangle$ means the full system's output state is either $|HH\rangle$ <u>or</u> $|HV\rangle$. The probability of getting $|HH\rangle$ output is

$$\left| \langle HH| \left(\frac{1}{\sqrt{2}} (|HH\rangle + |VV\rangle) \right) \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} \left(\langle HH|HH\rangle + \langle HH|VV\rangle \right) \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1+0) \right|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

The prob for $|HV\rangle$ comes out $= 0$.

So $P(|H\rangle)$ for Bob $= \frac{1}{2} + 0 = \frac{1}{2}$.

Doing a similar exercise for $|V\rangle$, we again get $\frac{1}{2}$.

So 11. option C works.

12. $|000\rangle \rightarrow |000\rangle$

$|001\rangle \rightarrow |001\rangle$

$|010\rangle \rightarrow |010\rangle$

$|011\rangle \rightarrow |011\rangle$

$|100\rangle \rightarrow |100\rangle$

$\left.\begin{array}{l} |101\rangle \rightarrow |110\rangle \\ |110\rangle \rightarrow |101\rangle \end{array}\right\}$ only times something happens.

$|111\rangle \rightarrow |111\rangle$

$$
\begin{array}{cccccccc}
|000\rangle & |001\rangle & |010\rangle & |011\rangle & |100\rangle & |101\rangle & |110\rangle & |111\rangle \\
\end{array}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

So option B.

(13) $|H\rangle \xrightarrow{\alpha} \cos\alpha\,|H\rangle + \sin\alpha\,|V\rangle$

This happens with probability

$$\left|\left(\cos\alpha\langle H| + \sin\alpha\langle V|\right)|H\rangle\right|^2$$

$$= \cos^2\alpha$$

(In the remaining $1-\cos^2\alpha$ probability, the incoming photon is absorbed.)

~~The out~~

Now $\cos\alpha|H\rangle + \sin\alpha|V\rangle$ goes through the β polarizer.

$$\cos\alpha|H\rangle + \sin\alpha|V\rangle \longrightarrow \cos\beta|H\rangle + \sin\beta|V\rangle.$$

This process happens with probability:

$$\left|\left(\cos\beta\langle H| + \sin\beta\langle V|\right)\left(\cos\alpha|H\rangle + \sin\alpha|V\rangle\right)\right|^2$$

$$= \left|\cos\beta\cos\alpha + \sin\beta\sin\alpha\right|^2$$

$$= \left|\cos(\alpha-\beta)\right|^2$$

$$= \cos^2(\alpha-\beta)$$

The probability of a photon making it to the detector is

$$\boxed{\cos^2\alpha \cos^2(\alpha-\beta)}$$

So option E.

(Note $\cos(\alpha-\beta) = \cos(\beta-\alpha)$.
You can see this by starting at
$\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta) \mp \sin\alpha\sin\beta$.)