Question 1

(a) Given that ket $|1\rangle$ is represented by a column vector $\begin{bmatrix}0\\1\end{bmatrix}$ and likewise ket $|0\rangle$ is represented by $\begin{bmatrix}1\\0\end{bmatrix}$, write the following states as column vectors.

(i) $|\psi_1\rangle = a|0\rangle + ib|1\rangle$.

(ii) $|\psi_2\rangle = ib|0\rangle + a|1\rangle$.

(iii) $|\psi_3\rangle = |0\rangle$.

(iv) $|\psi_4\rangle = a|1\rangle$.

(b) For the quantum states written as vectors above, find the corresponding bras as row vectors.

(c) Find the following inner products using matrix notation.

(i) $\langle\psi_1|\psi_1\rangle$.

(ii) $\langle\psi_2|\psi_4\rangle$.

(iii) $\langle\psi_1|\psi_2\rangle$.

(d) Given the following quantum states:

(i)

$$|\psi_a\rangle = \sqrt{\frac{1}{2}}a|0\rangle + \sqrt{\frac{1}{2}}ib|1\rangle .$$

(ii)

$$\langle\psi_b| = \sqrt{\frac{1}{3}}\langle 0| + \sqrt{\frac{2}{3}}i\langle 1| .$$

Find the probability amplitudes (the coefficients) of their conjugate vectors, i.e $\langle\psi_a|$ and $|\psi_b\rangle$.

(e) Calculate:

(i) $\langle\psi_b|\psi_a\rangle$.

(ii) $\langle\psi_a|\psi_b\rangle$.

Question 2

So far we have worked with quantum states using bra-ket notation, now we would like to acquaint ourselves with another powerful method of representing quantum states using vectors. You will soon realise how this simplifies our calculations and also allows us to carry out computations.

(a) Write the column vector for the following state:

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,,$$

where $a, b \in \mathbb{C}$

(b) How will you represent the the complex conjugate of $|\psi\rangle$?

(c) Given the following transformation:

$$\begin{aligned} |0\rangle &\to a\,|0\rangle + ib\,|1\rangle \\ |1\rangle &\to ib\,|0\rangle + a\,|1\rangle\,, \end{aligned}$$

how will you represent this as a matrix?

(d) A general 2-D matrix that rotates the basis states $|0\rangle$ and $|1\rangle$ through an angle θ is given by:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

How will the basis transform under this transformation matrix?

(e) Use matrix multiplication, how will a basis state $|0\rangle$ transform under the operation in c part?

(f) Given that $\theta = \frac{\pi}{2}$, how will $|0\rangle$ transform under the rotation matrix you wrote in (d) part?

(g) Let's say that you rotate $|1\rangle$ by an angle $\theta = \frac{\pi}{2}$ and then perform the transformation given in (c) part. Write the transformed state. Would the output change if we reversed the order of transformation?

Question 3

This question intends to make you more comfortable with the use of matrices to represent vectors and operators in larger quantum spaces, especially those that we encounter when dealing with composite systems.

Suppose we have two qubits. We label one of them qubit A, and the other qubit B. The quantum system for A has the basis $|0_A\rangle\,, |1_A\rangle$. The quantum system for B has the basis $|0_B\rangle\,, |1_B\rangle$.

(a) Let us first look at how to combine vectors from both spaces. Say $|\psi\rangle = a\,|0_A\rangle + b\,|1_A\rangle$, and $|\phi\rangle = c\,|0_B\rangle + d\,|1_B\rangle$. We combine the two vectors using the direct product (a.k.a. tensor product): $\otimes$. All we need to know is that it distributes as we expect.

$$\begin{aligned}|\psi\rangle \otimes |\phi\rangle &= (a\,|0_A\rangle + b\,|1_A\rangle) \otimes (c\,|0_B\rangle + d\,|1_B\rangle)\\ &= ac\,|0_A\rangle\,|0_B\rangle + ad\,|0_A\rangle\,|1_B\rangle + bc\,|1_A\rangle\,|0_B\rangle + bd\,|1_A\rangle\,|1_B\rangle\,.\end{aligned}$$

Note that we have removed the $\otimes$ symbol: this is for convenience. We can also remove the labels A, B and just keep the convention that the first ket is from A and the second from B. So our answer becomes

$$ac\,|0\rangle\,|0\rangle + ad\,|0\rangle\,|1\rangle + bc\,|1\rangle\,|0\rangle + bd\,|1\rangle\,|1\rangle\,.$$

Since there are four basis vectors now, our column vectors will have four entries. If

$$\begin{aligned}|0\rangle\,|0\rangle &= \begin{bmatrix}1\\0\\0\\0\end{bmatrix},\\ |0\rangle\,|1\rangle &= \begin{bmatrix}0\\1\\0\\0\end{bmatrix},\\ |1\rangle\,|0\rangle &= \begin{bmatrix}0\\0\\1\\0\end{bmatrix},\\ |1\rangle\,|1\rangle &= \begin{bmatrix}0\\0\\0\\1\end{bmatrix},\end{aligned}$$

express $|\psi\rangle \otimes |\phi\rangle$ as a column vector.

(b) Once we have the ket's matrix representation, we find the bra's representation. For a ket represented by the column

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix},$$

the corresponding bra is represented by

$$\begin{bmatrix}a^*, b^*, c^*, d^*\end{bmatrix}.$$

Use this to find the inner product of $|\psi\rangle \otimes |\phi\rangle$ with itself.

(c) We now turn to writing out the matrix representation of some operators that act on this composite system. Consider the single qubit operation, B, defined below.

$$|0\rangle \xrightarrow{B} \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$
$$|1\rangle \xrightarrow{B} \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle .$$

Suppose it acts on both qubits independently. Find the single matrix representing this combined operation.

(d) Consider now a controlled gate shown in the figure. If the first qubit is in the state $|0\rangle$, the second qubit subject to the single qubit operation defined in part c. If the first qubit is in the state $|1\rangle$, nothing happens to the second qubit. In either case, the first qubit is unaffected. Write out the logic/truth table for this controlled gate (as in, tell us what this controlled gate does to the four basis states). Use this to write out its matrix.

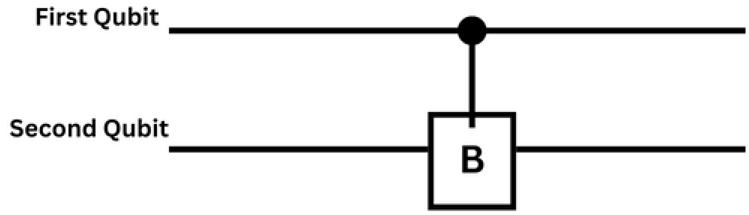

Figure 1: A Controlled Gate

(e) Use matrix multiplication to find the output state for the setup above if the input state is $|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Question 4

Using the math practised above, you can now solve quantum state progression questions using matrices and vectors. Using the interferometer figure below 2, solve the following. All the beam splitters are identical and their action is defined below.

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Solve the question using matrix notation ONLY.

(a) Describe the state progression for an input state $|1\rangle$.

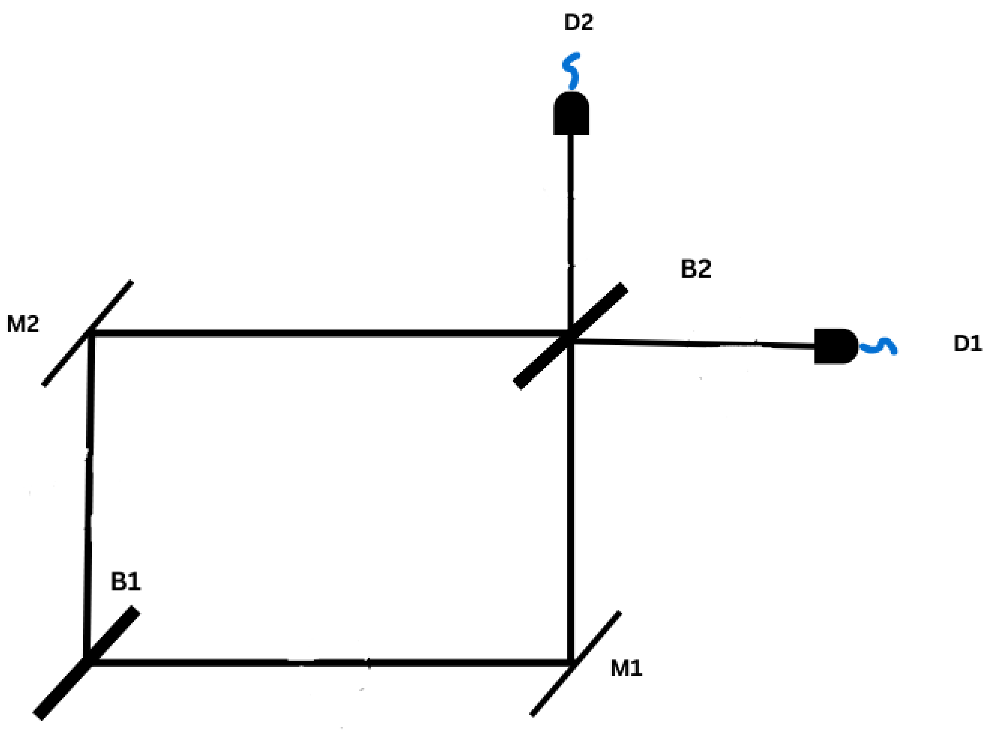

Figure 2: Mach-Zehnder interferometer

(b) What are the probabilities of $D1$ and $D2$ clicking?

(c) If a device is placed in the vertical arm before the second beam splitter B2 and it has the following action

$$|1\rangle \rightarrow i\,|1\rangle$$

calculate how the probabilities change for each detector.

Question 5

In this question we will investigate quantum erasure using a Mach-Zehnder interferometer with polarizers on its various arms as shown in 3. You will discover that depending on the polarizer we use, the photon interference can be altered. A quantum state can be described by its polarization. The horizontally and vertically polarized photon can be represented as $|H\rangle$ and $|V\rangle$ respectively. The diagonal and anti-diagonal polarized states are written in $|H\rangle$ and $|V\rangle$ basis as:

$$|D\rangle = \frac{1}{\sqrt{2}}\,|H\rangle + |V\rangle$$

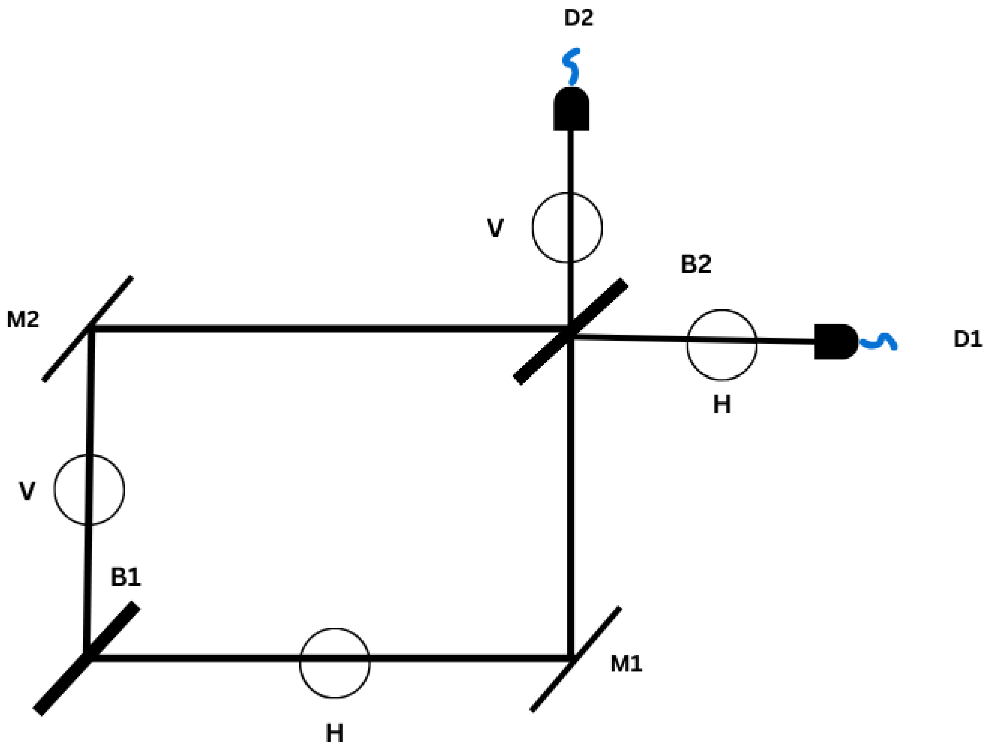

Figure 3: Mach-Zehnder interferometer

$$|A\rangle = \frac{1}{\sqrt{2}} |H\rangle - |V\rangle$$

Given that we input a composite state given by $|1\rangle \otimes |A\rangle$,

(a) Describe the state progression.

(b) What is the probability of $D1$ and $D2$ clicking?

(c) Are these probabilities different from the ones we calculated for a simple interferometer without polarizers? Explain.

(d) In this part, we will replace the horizontal and vertical polarizers right before the detectors with anti-diagonal polarizers. Repeat the previous parts with this change in the interferometer.

(e) Compare your answer with c part and explain how this is an example of quantum erasure.

Question 6

This question explores the Quantum Zeno Effect. See the corresponding diagram. The input photon passes through N rotation gates, each labeled θ. The action of each rotation gate is defined below.

$$|H\rangle \xrightarrow{\theta} \cos(\theta)|H\rangle + \sin(\theta)|V\rangle$$
$$|V\rangle \xrightarrow{\theta} -\sin(\theta)|H\rangle + \cos(\theta)|V\rangle$$

where $\theta \in \mathbb{R}$.

(a) Write out the matrix representing the rotation gate in the $|H\rangle, |V\rangle$ basis.

(b) We now intend to find the single matrix representing the action of all of the N rotation gates. To start, consider two different rotation gates, one with $\theta = \phi_0$ and another with $\theta = \phi_1$. Use matrix multiplication to find the single matrix representing the action of these two gates. You should be able to conclude that this combined action is equivalent to one phase gate with $\theta = \phi_0 + \phi_1$.

(c) The above result allows us to immediately find the matrix representing the action of two rotation gates, both with $\theta = \phi$. Write it.

(d) Use the two foregoing parts to find the matrix representing the action of all N rotation gates, all with $\theta = \phi$. You should be able to conclude that this combined action is equivalent to one phase gate with $\theta = N\phi$.

(e) We input a photon with polarisation $|H\rangle$ and set $\theta = \frac{\pi}{2N}$. The detector at the end, D, detects photons in the polarisation state $|H\rangle$. What is the probability that the detector clicks?

(f) We now place horizontal polarizers after every rotation gate. This is us "monitoring" the system. Consider the first time the photon passes through the rotation gate and passes through the polariser. What is the probability that the photon makes it to the second rotation gate? What state will the photon be in right before entering the second gate?

(g) Find the probability, in terms of N, that the detector clicks now.

We hope your probability comes out to be $\cos^{2N}(\frac{\pi}{2N})$. It is clear that, as long as the cosine does not output zero, this probability is *non-zero*! Repeatedly monitoring the system seems to slow down its evolution. .

(h) We now want to show that doing more measurements enhances this effect. To do so, we want to show that as $N \to \infty$, our probability approaches 1. We will only motivate this here. Since $\theta(= \frac{\pi}{2N}) \to 0$ as $N \to \infty$, using the Maclaurin series for $\cos^{2N}(\theta)$ seems fair. Write out the first few terms of the expansion, and plug in $\theta = \frac{\pi}{2N}$ at the end. What does your expression approach as $N \to \infty$?

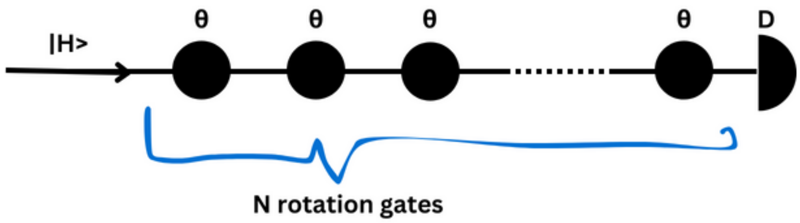

Figure 4: Setup

Question 7

Consider the quantum circuit shown. Only if the first qubit and second qubit are *both* in the state $|1\rangle$, the NOT gate acts on the third qubit. Nothing happens to the first two qubits. This is known as the Toffoli gate. The action of the NOT gate is described below.

$$|0\rangle \xrightarrow{\text{NOT}} |1\rangle$$
$$|1\rangle \xrightarrow{\text{NOT}} |0\rangle\,.$$

(a) Write the truth table for this setup (as in, tell us what this setup does to the basis states).

(b) Use this table to write out the matrix for this setup.

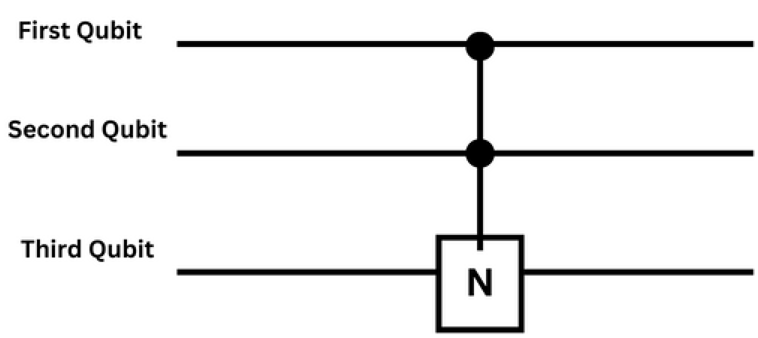

Figure 5: Toffoli Gate

Question 8

Consider the quantum circuit shown. The action of the H gate (the Hadamard gate) is shown below.

$$|0\rangle \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|1\rangle \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

N represents the NOT gate. It's being controlled- only if the first qubit is in the state $|1\rangle$ does it act. Nothing happens to first qubit.

(a) Given that we input the state $|0\rangle|0\rangle$, find the state at the point indicated with the arrow. (Note that we have dropped the $\otimes$ symbol ; this we do for ease. Note also that we are using position to tell which ket describes which qubit: the first ket from the left is for the first qubit, the second one for the second.)

(b) Suggest some combination of gates for the dotted box such that the output state is also $|0\rangle|0\rangle$.

(c) Check whether or not your suggested gates do this job (as in, recoup the input state) for *any* input state.

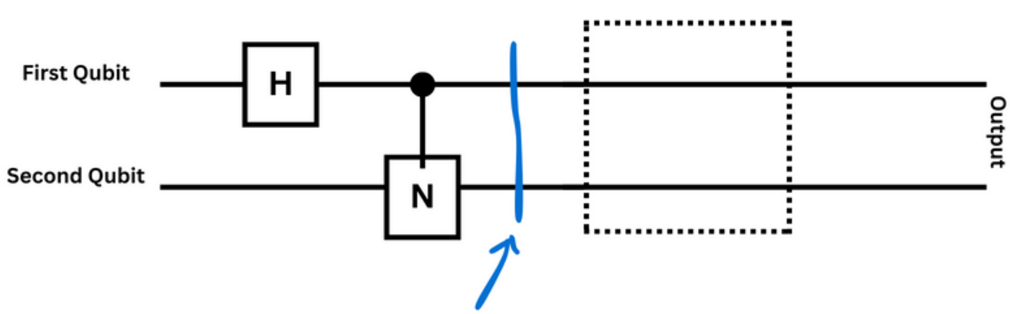

Figure 6: Quantum Circuit for Question 5