Question 1

Identify the photon polarization states $|H\rangle$, $|V\rangle$, $|L\rangle$, $|R\rangle$, $|D\rangle$, $|A\rangle$ and $\frac{|D\rangle-|A\rangle}{\sqrt{2}}$ on a Bloch sphere, where

$$|D\rangle = \frac{1}{\sqrt{2}}\Big(|H\rangle + |V\rangle\Big)$$
$$|A\rangle = \frac{1}{\sqrt{2}}\Big(|H\rangle - |V\rangle\Big)$$
$$|L\rangle = \frac{1}{\sqrt{2}}\Big(|H\rangle + i\,|V\rangle\Big)$$
$$|R\rangle = \frac{1}{\sqrt{2}}\Big(|H\rangle - i\,|V\rangle\Big).$$

An arbitrary polarisation state of a photon can be represented on the Bloch sphere as

$$\cos\left(\frac{\theta}{2}\right)|H\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\,|V\rangle\,.$$

($|H\rangle$ takes the place of $|0\rangle$, and $|V\rangle$ takes the place of $|1\rangle$ from our discussions in class).

Question 2

Given a qubit in the $|0\rangle$ state, living on the Bloch sphere. Write down the resultant state if we do a 2π rotation about the x-axis. What is the resultant state if we do a 4π rotation instead?

Question 3

In this question we will investigate how rotations performed on the qubits living on a Bloch sphere can help us achieve the operations defined by various gates.

(a) Show that a $\frac{\pi}{2}$ rotation about the y-axis is not a Hadamard gate. For this you will need to choose a basis state and apply the standard operation of the Hadamard gate defined below:

$$|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Hint: You need to work with $|+z\rangle\, and\, |-z\rangle$ basis defined on the Bloch sphere.

(b) Show that a π rotation about the y-axis is not a Not gate. The operation of the Not gate is defined as follows:

$$\begin{aligned}|0\rangle &= |1\rangle \\ |1\rangle &= |0\rangle\,.\end{aligned}$$

(c) Show that a $\left(\frac{\pi}{2}\right)_y$ followed by $(\pi)_x$ rotation achieves a Hadamard gate. The alphabets in the subscripts represent the axis of rotation.

(d) Show that a π rotation about the $\frac{e_x+e_z}{\sqrt{2}}$ achieves a Hadamard gate.

Question 4

We have a qubit in the state $|0\rangle$, and we wish to put it in the state $|1\rangle$. One way to do so is to subject our qubit to a π rotation about the y-axis.

(a) Show that such a rotation does what we intend.

Manipulation of real-world qubits isn't so straightforward. Suppose subjecting our qubit to a θ rotation about any axis actually rotates our qubit by $\theta(1-\epsilon)$ radians about the same. All we know about ϵ is that $0 < \epsilon \ll 1$ such that ϵ^5 and all higher powers can be ignored.

(b) How well does our original idea do now? To calculate this, calculate the modulus square of the overlap of $|1\rangle$ with the output from the rotation.

(c) Consider now a sequence of rotations. First, subject the qubit in $|0\rangle$ to a $\frac{\pi}{2}$ rotation about the y-axis. Then, subject the resultant state to a π rotation about the x-axis. Finally, subject it to a $\frac{\pi}{2}$ rotation about the y-axis. Calculate how well this sequence does in achieving $|1\rangle$, as done in part b.

(d) Which method performs better?

Hint: Think Taylor Series!

Question 5

Suppose we have a qubit in $|0\rangle$.

(a) We subject it to a $\frac{\pi}{2}$ rotation about the y-axis. We then rotate it by the same angle about the z-axis. Finally, we subject it to a rotation about the x-axis by the same angle. What is the final state of the qubit? Try this just by looking at the Bloch sphere and also by explicit calculation.

(b) Again, consider a a qubit in $|0\rangle$. We subject it to a $\frac{\pi}{2}$ rotation about the y-axis. We then rotate it by $\frac{\pi}{4}$ about the z-axis. Finally, we subject it to a rotation about the x-axis by $\frac{\pi}{2}$. What is the final state of the qubit? Try this just by looking at the Bloch sphere, and also by explicit calculation.

Do you notice something odd about the final states, especially when you compare it with what you expected just by looking at the Bloch sphere?

Question 6

We now try to prove that

$$e^{i\theta\hat{A}} = \cos(\theta)\mathbb{1} + i\sin(\theta)\hat{A}$$

for $\hat{A}$ such that $\hat{A}^2 = \mathbb{1}$ and $\theta \in \mathbb{R}$.

(a) We define $e^{\hat{B}}$ by the power series

$$\sum_{n=0}^{\infty} \frac{\hat{B}^n}{n!} = \mathbb{1} + \hat{B} + \frac{\hat{B}^2}{2!} + \frac{\hat{B}^3}{3!} + \frac{\hat{B}^4}{4!} + \cdots$$

Write out the first few (at least the first four) terms of the expansion for $e^{i\theta\hat{A}}$.

(b) Write out the first few (at least the first two) terms of the Maclaurin series for $\cos(\theta)$ and $\sin(\theta)$.

(c) Compare parts a and b to conclude that (at least to the order considered), the claim we want to prove is true.

Hint: you must use that $\hat{A}^2 = \mathbb{1}$!

(d) Use what we learned in Homework 1, Question 8 to show that the result holds even if we consider all terms in the expansions.

(e) Food for thought: what does it mean to have an infinite sum of operators, as we have in part a? Can we define some notion of convergence of such sums? Such considerations tell us why a Hilbert Space (what we have been calling a quantum space) is not just any vector space with an inner product; it must meet other conditions, too! These questions fall much beyond the scope of this course, but you are welcome to explore them.

Question 7

In class, we said that the solution to

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

is

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t} |\psi(0)\rangle .$$

Here $\hat{H}$ is the Hamiltonian, $|\psi(0)\rangle$ is the initial state of the system, and $|\psi(t)\rangle$ is the state of system at time t.

Show that our form of $|\psi(t)\rangle$ satisfies the differential equation. For this question, you may treat $\hat{H}$ as a number (of course, this is not true! If you wish, try this problem without this assumption. You might hit a snag when trying to figure out $\frac{d}{dt}e^{-\frac{i}{\hbar}\hat{H}t}$. Try using the limit definition here. This is an extra exercise, and falls outside the scope of this course).

Question 8

Consider a spin-1/2 particle in a constant magnetic field in the x-direction. The Hamiltonian is

$$\hat{H} = b\hbar\hat{X}.$$

(a) Write the matrix for $\hat{H}$.

(b) If the system is initially in the state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, find its state at time t. Do any measurement probabilities change with time?

(c) Repeat part b with initial state $|0\rangle$. Do the probabilities of measuring $|0\rangle$ and $|1\rangle$ change with time? If so, find them (as functions of time).