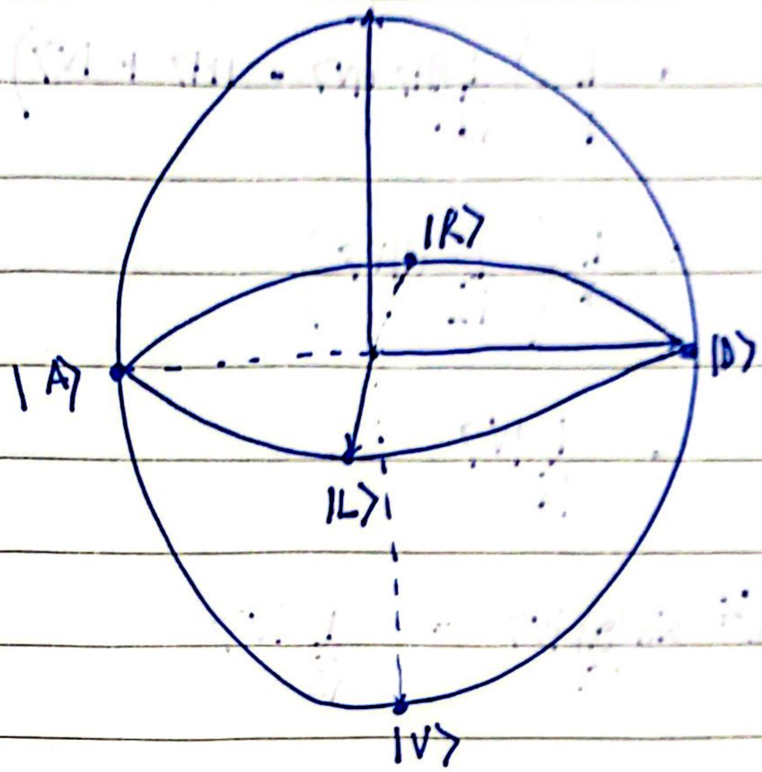
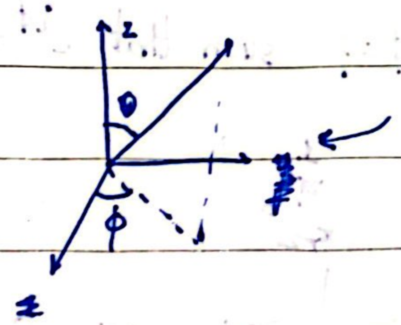
Question 1

Date



where



An arbitrary state is represented using the formula:

$$\cos\left(\frac{\theta}{2}\right)|H\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|V\rangle$$

! pluggin $\cancel{\frac{\theta}{2} = \frac{\pi}{3}}$ $\theta = 0$ and $\phi = 0$ gives us $|H\rangle$

Similarly we can all the states, for example

! $|L\rangle$ by pluggin $\theta = \pi/2$ and $\phi = \pi/2$

gives me $\cos\left(\frac{\pi}{4}\right)|H\rangle + e^{i\pi/2}\sin\left(\frac{\pi}{4}\right)|V\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle + i|V\rangle\right)$

! $|D\rangle = \cos\left(\frac{\pi}{4}\right)|H\rangle + e^{i0}\sin\left(\frac{\pi}{4}\right)|V\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle + |V\rangle\right)$

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$$\frac{|D\rangle - |A\rangle}{\sqrt{2}} = \frac{1}{2}\left(|H\rangle + |V\rangle - |H\rangle + |V\rangle\right)$$
$$= |V\rangle.$$

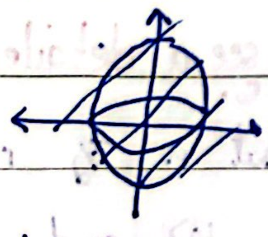
~~For a~~ use general state on a Bloch Sphere formula..

$$\cos\left(\frac{\theta}{2}\right)|H\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|V\rangle = |V\rangle$$

Setting $\cancel{\frac{\theta}{2}}$. $\theta = \cancel{\frac{\pi}{2}}\ \pi$

We get, $e^{i\phi}|V\rangle = |V\rangle$ $\phi = 0$

$|V\rangle = |V\rangle \rightarrow$ Therefore, on the Bloch



Shere $|V\rangle$ can be labelled as $\frac{|D\rangle - |A\rangle}{\sqrt{2}}$.

Question 2

Doing a 2π rotation about the X-axis gives ~~the~~ us the Hamiltonian $\hat{H} = \frac{\hbar\omega}{2}\hat{X}$

$$|\Psi(t)\rangle = \hat{U}|\Psi(0)\rangle$$

$$\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}} = e^{-i\frac{\hbar\omega X}{2}\frac{t}{\hbar}} = e^{-i\frac{\omega t}{2}\hat{X}}$$

$$= \cos\left(\frac{\omega t}{2}\right)\mathbb{1} - i\sin\left(\frac{\omega t}{2}\right)\hat{X}$$

$$\omega t = 2\pi$$

$$|\Psi(t)\rangle = \left(\cos(\pi)\mathbb{1} - i\sin(\pi)\hat{X}\right)|0\rangle$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -|0\rangle$$

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If we did a 4π rotation

$$|\psi(t)\rangle = \left(\cos(2\pi)\,\mathbb{1} - i\sin(2\pi)\,\hat{X}\right)|0\rangle$$

$$= \mathbb{1}\,|0\rangle$$

$$= |0\rangle$$

We land on the state, ~~just that~~ but the state has picked up a phase factor.

Question 3 * We will see a detailed reason for this in part c.

(a) You can check that a $\pi/2$ rotation about the y-axis of the state $|0\rangle$ and $|1\rangle$ will give us the desired result needed to achieve the Hadamard gate. However, it won't work for the superposition states like.

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

$$|\psi(0)\rangle \xrightarrow{H\text{-gate}} \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) + \frac{i}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right\}$$

$$= \frac{1}{2}\left(|0\rangle + |1\rangle + i|0\rangle - i|1\rangle\right)$$

Let's see if we get this state by

$-\left(\frac{\pi}{2}\right)_y$

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$$\hat{\mathcal{U}} = e^{-i\frac{\omega t}{2}\hat{Y}}$$

$$= \cos\left(\frac{\omega t}{2}\right)\mathbb{1} - i\sin\left(\frac{\omega t}{2}\right)\hat{Y} \qquad \omega t = \frac{\pi}{2}$$

$$= \frac{1}{\sqrt{2}}\mathbb{1} - \frac{i}{\sqrt{2}}\hat{Y}$$

$|\psi(t)\rangle =$

$$\left(\frac{1}{\sqrt{2}}\mathbb{1} - \frac{i}{\sqrt{2}}\hat{Y}\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right).$$

$$\frac{1}{2}|0\rangle + \frac{i}{2}|1\rangle - \frac{i}{2}\hat{Y}|0\rangle + \frac{i}{2}\hat{Y}|1\rangle$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$$

$$\sigma_y|0\rangle = i|1\rangle$$

$\sigma_y|0\rangle$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} \qquad \sigma_y|1\rangle = -i|0\rangle$$

$$|\psi(t)\rangle = \frac{1}{2}|0\rangle + \frac{i}{2}|1\rangle + \frac{|1\rangle}{2} + \frac{1}{2}|0\rangle$$

$$\frac{1}{2}\left(|0\rangle + |1\rangle + i|0\rangle - i|1\rangle\right) \neq \frac{1}{2}\left(|0\rangle + i|1\rangle + |1\rangle + |0\rangle\right)$$

* Hence, this is sufficient to prove that it's not a Hadamard gate.

→ Read the discussion in part c!

(C) Achieving a Hadamard gate Date

$$\left(\frac{\pi}{2}\right)_y = \hat{U}_y \qquad (\pi)_x = \hat{U}_x$$

$$|\psi(t)\rangle = \hat{U}_x \hat{U}_y |\psi(0)\rangle$$

$$\hat{U}_y = \cancel{e^{-i\pi/4}} \quad e^{-i\frac{\omega t}{2}\hat{Y}} \qquad \hat{U}_x = e^{-i\frac{\omega t}{2}\hat{X}}$$

Let's see how this state looks for $|\psi(0)\rangle = |0\rangle$

$$\hat{U}_y|0\rangle = \left(\cos\left(\frac{\pi}{4}\right)\mathbb{1} - i\sin\left(\frac{\pi}{4}\right)\hat{Y}\right)|0\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}\left(i|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{|1\rangle}{\sqrt{2}}$$

$$\hat{U}_x\left(\hat{U}_y|0\rangle\right) = \hat{U}_x\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$$

$$= \left(\cos\left(\frac{\pi}{2}\right)\mathbb{1} - i\sin\left(\frac{\pi}{2}\right)\hat{X}\right)\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$$

$$= -i\hat{X}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$$

$$= -i\left(\frac{|1\rangle + |0\rangle}{\sqrt{2}}\right)$$

a global phase of ~~the~~ $-i$ picked up but we do achieve the Hadamard gate for $|0\rangle$.

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Similarly, we can do for $D|de\rangle$ state.

$$e^{i}\left(\cos\left(\frac{\pi}{4}\right)\mathbb{1} + i\sin\left(\frac{\pi}{4}\right)\hat{Y}\right)|1\rangle$$

$$e^{i}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-i\\0\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{2}}\hat{Y}|1\rangle$$

$$\frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{2}}\left(-i|0\rangle\right)$$

$$\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle \quad -1\left(\frac{1}{\sqrt{2}}|0\rangle - |1\rangle\right)$$

global phase

Now apply $(\pi)_x$

$$\left(\cos\left(\frac{\pi}{2}\right)\mathbb{1} - i\sin\left(\frac{\pi}{2}\right)\hat{X}\right)\, -1\left(\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right)$$

$$= \frac{i}{\sqrt{2}}\sin\left(\frac{\pi}{2}\right)\hat{X}\left(|0\rangle - |1\rangle\right)$$

$$= \frac{i}{\sqrt{2}}\left(|1\rangle - |0\rangle\right) = \frac{-i\left(|0\rangle - |1\rangle\right)}{\sqrt{2}}$$

a global phase of $-i$

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Observe that $|1\rangle$ and $|0\rangle$ pick up the same global phase, therefore, the Hadamard gate will work for any superposition state.

However, we can see that only $(\pi/2)y$ rotation will lead to different phase factors being picked up by the basis states.

(d) $\dfrac{e_x + e_z}{\sqrt{2}} = \dfrac{\hat{X}+\hat{Z}}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\hat{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\mathcal{U}_{\frac{e_x+e_z}{\sqrt{2}}} = \cos\dfrac{\theta}{2}\,\mathbb{1} - i\sin\dfrac{\theta}{2}\,\hat{A}$ plugging $\theta = \pi$

$$= -i\hat{A}$$

$$\mathcal{U}_{\frac{e_x+e_z}{\sqrt{2}}}|0\rangle = \cancel{\frac{-i\cdot 1}{\sqrt{2}}} \; \frac{-i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{-i}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{-i}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\mathcal{U}_{\frac{e_x+e_z}{\sqrt{2}}}|1\rangle = \frac{-i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-i}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Again, same phase factor picked up ~~the~~.

(b) $(\pi)_y = e^{-i\frac{\omega t}{2}\hat{Y}}$, where $\omega t = \pi$

$= \cos\left(\frac{\pi}{2}\right)\mathbb{1} - i\sin\left(\frac{\pi}{2}\right)\hat{Y}$

$= -i\hat{Y}$

$|\psi(t)\rangle = -i\hat{Y}|0\rangle$

$= -i(i|1\rangle)$

$= |1\rangle$

$|\psi(t)\rangle = -i\hat{Y}|1\rangle$

$= -i(-i|0\rangle)$

$= -|0\rangle$

~~The~~ We do achieve a Not-gate for the two basis states. However, since the state $|1\rangle$ picks up a phase factor of -1 while the $|0\rangle$ state does not. Therefore, this gate will not work for superposition states.

~~(c)~~

Q4.
a) $(\pi)_y \rightarrow e^{-i\frac{\pi}{2}\hat{Y}} = \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\hat{Y}$

$= -i\hat{Y}$

$-i\hat{Y}|0\rangle = -i(i|1\rangle) = |1\rangle$, as desired.

Note that $\hat{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

so $\hat{Y}|0\rangle = i|1\rangle$ and $\hat{Y}|1\rangle = -i|0\rangle$.

b) $(\pi)_y \xrightarrow{\text{in practice}} e^{-i\frac{\pi}{2}(1-\varepsilon)\hat{Y}}$

$= \cos\left(\frac{\pi}{2}(1-\varepsilon)\right) - i\sin\left(\frac{\pi}{2}(1-\varepsilon)\right)\hat{Y}$

acting this on $|0\rangle$ gives

$\cos\left(\frac{\pi}{2}(1-\varepsilon)\right)|0\rangle - i\sin\left(\frac{\pi}{2}(1-\varepsilon)\right)(i|1\rangle)$.

$= \cos\left(\frac{\pi}{2}(1-\varepsilon)\right)|0\rangle + \sin\left(\frac{\pi}{2}(1-\varepsilon)\right)|1\rangle$

since $i^2 = -1$.

How well does it do?

$\left|\langle 1|\left(\cos\left(\frac{\pi}{2}(1-\varepsilon)\right)|0\rangle + \sin\left(\frac{\pi}{2}(1-\varepsilon)\right)|1\rangle\right)\right|^2$

$= \sin^2\left(\frac{\pi}{2}(1-\varepsilon)\right)$. since $\langle 1|0\rangle = 0$ and $\langle 1|1\rangle = 1$.

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c) $|0\rangle \xrightarrow{\left(\frac{\pi}{2}\right)_y} \underbrace{e^{\frac{-i\pi}{4}\hat{y}(1-\varepsilon)}}\ |0\rangle$

$$\left[\cos\left(\frac{\pi}{4}(1-\varepsilon)\right) + i\sin\left(-\frac{\pi}{4}(1-\varepsilon)\right)\hat{y}\right]|0\rangle$$

$\sin(a) = -\sin(-a)$
and
$\hat{y}|0\rangle = i|1\rangle$.

$$= \cos\left(\frac{\pi}{4}(1-\varepsilon)\right)|0\rangle + \sin\left(\frac{\pi}{4}(1-\varepsilon)\right)|1\rangle.$$

The $(1-\varepsilon)$ keeps tagging along. When writing angles now, I omit it with the understanding that at the end, I write it again. This makes the working less cumbersome.

So, we now act with $(\pi)_x \rightarrow e^{\frac{-i\pi}{2}\hat{x}}$ on

$$\cos\left(\frac{\pi}{4}\right)|0\rangle + \sin\left(\frac{\pi}{4}\right)|1\rangle.$$

$e^{-\frac{i\pi}{2}\hat{x}} = \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\hat{x}$, so we have

$$\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)|0\rangle + \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)|1\rangle$$
$$-i\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)|1\rangle - i\sin\left(\frac{\pi}{2}\right)\left(\frac{\pi}{4}\right)|0\rangle.$$

$$= \left(\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)\right)|0\rangle$$
$$+$$
$$\left(\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)\right)|1\rangle.$$

We now implement our final gate

$$\left(\frac{\pi}{2}\right)_y \longrightarrow \underbrace{e^{\frac{-i\pi}{4}\hat{Y}}}_{=\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\hat{Y}} \text{ on this state}$$

This yields

$$\left[\cos^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)\right]|0\rangle$$

$$+$$

$$\left[\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) - i\cos^2\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right)\right]|1\rangle$$

$$+$$

$$\left[\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - i\sin^2\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right)\right]|1\rangle$$

$$+$$

$$\left[-\sin^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)\right]|0\rangle.$$

The overlap with $|1\rangle$ gives

$$2\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{2}\right)\left[\cos^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{\pi}{4}\right)\right]$$

$\cancel{= 2\cos\left(\frac{\pi}{4}\right)}$ Use $2\sin a\cos a = \sin(2a)$

$$= \sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)$$

$$\left|\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right|^2 = \sin^2\left(\frac{\pi}{2}\right)\cos^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right)$$

$$= \sin^2\left(\frac{\pi}{2}\right)\left[1-\sin^2\left(\frac{\pi}{2}\right)\right] + \sin^2\left(\frac{\pi}{2}\right)$$

$$= 2\sin^2\left(\frac{\pi}{2}\right) - \sin^4\left(\frac{\pi}{2}\right).$$

Now I bring the $(1-\varepsilon)$ back:

$$2\sin^2\left(\frac{\pi}{2}(1-\varepsilon)\right) - \sin^4\left(\frac{\pi}{2}(1-\varepsilon)\right).$$

d) I write out the Taylor series for both results. Here we exploit the fact that $0<\varepsilon \ll 1$ such that $\varepsilon^5, \varepsilon^6, \varepsilon^7, \dots \approx 0$.

With one step:

$\sin^2\left(\frac{\pi}{2}(1-\varepsilon)\right)$. So we'd like $\sin^2(\theta)$ near $\theta = \frac{\pi}{2}$ so I write out the Taylor series for $\sin^2(\theta)$ at $\frac{\pi}{2}$.

$$\sin^2\theta \approx 1 - \frac{2}{2!}\left(\theta-\frac{\pi}{2}\right)^2 + \frac{8}{4!}\left(\theta-\frac{\pi}{2}\right)^4$$

$\theta = \frac{\pi}{2}(1-\varepsilon)$ so we get

$$1 \underbrace{- \frac{\pi^2\varepsilon^2}{4} + \frac{\pi^4\varepsilon^4}{24}}_{\text{Error due to } \varepsilon \neq 0}.$$

what our answer ideally should be

for our sequence of rotations:

$$2\left[1-\frac{\pi^2\varepsilon^2}{4}+\frac{\pi^4\varepsilon^4}{24}\right]-\sin^4\left(\frac{\pi}{2}(1-\varepsilon)\right).$$

so I now write the Taylor series for $\sin^4(\theta)$ near $\theta=\frac{\pi}{2}$.

$$\sin^4(\theta)\approx 1-\frac{4}{2!}\left(\theta-\frac{\pi}{2}\right)^2+\frac{40}{4!}\left(\theta-\frac{\pi}{2}\right)^4$$

with our θ, this yields

$$1-2\left(-\frac{\pi}{2}\varepsilon\right)^2+\frac{50}{3}\left(-\frac{\pi}{2}\varepsilon\right)^4$$

$$1-\frac{2}{4}\pi^2\varepsilon^2+\frac{50}{3\cdot 2^4}\pi^4\varepsilon^4$$

Putting it all together, we get

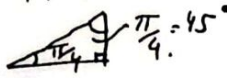
$$1+\frac{2\pi^4\varepsilon^4}{24}+\frac{50}{3\cdot 2^4}\pi^4\varepsilon^4$$

$$=1+\underbrace{O(\varepsilon^4)}_{\text{some error of power } \varepsilon^4.}$$

This method performs better since it has error of power ε^4, which is smaller than the ε^2 error ~~in~~ for the direct route (since $0<\varepsilon<1$).

Q5) a)

$$\left(\frac{\pi}{2}\right)_y \longrightarrow e^{-i\left(\frac{\pi}{4}\right)\hat{Y}} \qquad \cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = 1/\text{sqrt } 2$$



$$e^{-i\left(\frac{\pi}{4}\right)\hat{Y}}|0\rangle = \left[\cos\left(\frac{\pi}{4}\right)\mathbb{1} - i\sin\left(\frac{\pi}{4}\right)\hat{Y}\right]|0\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}\left(i|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right].$$

Note that $\hat{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, so

$$\hat{Y}|0\rangle = i|1\rangle.$$

Now, $\left(\frac{\pi}{2}\right)_z \longrightarrow e^{-i\left(\frac{\pi}{4}\right)\hat{z}} = \cos\left(\frac{\pi}{4}\right)\mathbb{1} - i\sin\left(\frac{\pi}{4}\right)\hat{z}$

$$= \frac{1}{\sqrt{2}}\mathbb{1} - \frac{i}{\sqrt{2}}\hat{z} = \frac{1}{\sqrt{2}}\left[\mathbb{1} - i\hat{z}\right].$$

Acting this on our ~~given~~ state found above gives

$$\left(\frac{1}{\sqrt{2}}\right)^2\left[\mathbb{1} - i\hat{z}\right]\left[|0\rangle + |1\rangle\right]$$

$$\frac{1}{2}\left[|0\rangle + |1\rangle - i|0\rangle + i|1\rangle\right]$$

$$\cancel{\frac{1}{2}\left[(|0\rangle\right.} \qquad \frac{1}{2}\left[(1-i)|0\rangle + (1+i)|1\rangle\right].$$

Now, $\left(\frac{\pi}{2}\right)_x \to e^{-i\left(\frac{\pi}{4}\right)\hat{x}}$

$$= \cos\left(\frac{\pi}{4}\right)\mathbb{1} - i\sin\left(\frac{\pi}{4}\right)\hat{x}$$

$$= \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\hat{x} = \frac{1}{\sqrt{2}}(1 - i\hat{x}).$$

(Note that I omit the $\mathbb{1}$, since $\mathbb{1}|\psi\rangle = |\psi\rangle$ $\hat{z}$ for all $|\psi\rangle$, $\frac{1}{\sqrt{2}}\mathbb{1}$ just multiplies the state by $\mathbb{1}$.)

Acting this on the state found above, we get

$$\frac{1}{2\sqrt{2}}\Big[(1-i)|0\rangle + (1+i)|1\rangle + \\ -i(1-i)|1\rangle - i(1+i)|0\rangle\Big].$$

Since $\hat{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so $\hat{x}|0\rangle = |1\rangle$ and $\hat{x}|1\rangle = |0\rangle$.

This simplifies to

$$\frac{1}{2\sqrt{2}}\left[(1-i)|0\rangle + (1-i)|0\rangle\right]$$

$$= \frac{\cancel{2}(1-i)|0\rangle}{\cancel{2}\sqrt{2}} = \frac{1-i}{\sqrt{2}}|0\rangle$$

$$= e^{-i\frac{\pi}{4}}|0\rangle.$$

I couldn't have guessed the global phase by just looking at the Bloch sphere rotations!

b) The first step is the same.

$$|0\rangle \to \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right].$$

Then

$$\left(\frac{\pi}{4}\right)_{\hat{z}} \to e^{\frac{-i\pi}{8}\hat{z}} = \cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right)\hat{z}$$

Acting this on the state above gives

$$\frac{1}{\sqrt{2}}\left[\cos\left(\frac{\pi}{8}\right)|0\rangle + \cos\left(\frac{\pi}{8}\right)|1\rangle - i\sin\left(\frac{\pi}{8}\right)|0\rangle + i\sin\left(\frac{\pi}{8}\right)|1\rangle\right].$$

$$= \frac{1}{\sqrt{2}}\left[\left(\cos\left(\frac{\pi}{8}\right) - i\sin\frac{\pi}{8}\right)|0\rangle + \left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right)|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{\frac{-i\pi}{8}}|0\rangle + e^{\frac{i\pi}{8}}|1\rangle\right].$$

Note $e^{\frac{\pm i\pi}{8}}$ is <u>not</u> a global phase!

Now, $\left(\frac{\pi}{2}\right)_x \to e^{\frac{-i\pi}{4}\hat{X}} = \frac{1}{\sqrt{2}}\left(1 - i\hat{X}\right).$

Acting this on the state above yields

$$\left(\frac{1}{\sqrt{2}}\right)^2\left[e^{-\frac{i\pi}{8}}|0\rangle + e^{+\frac{i\pi}{8}}|1\rangle - ie^{-\frac{i\pi}{8}}|1\rangle - ie^{i\pi/8}|0\rangle\right].$$

Q.6.

We can simplify:

$$\frac{1}{2} e^{-\frac{i\pi}{8}} \left[\left(1 - i e^{\frac{i\pi}{4}}\right)|0\rangle + \left(e^{\frac{i\pi}{4}} - i\right)|1\rangle \right].$$

$e^{-\frac{i\pi}{8}}$ is a global phase and can be dropped.

The final state is

$$\frac{1}{2} \left[\left(1 - i e^{\frac{i\pi}{4}}\right)|0\rangle + \left(e^{\frac{i\pi}{4}} - i\right)|1\rangle \right].$$

Please review this result carefully. If you see a mistake, let me know.

Q.6.

a) $$e^{i\theta\hat{A}} = \mathbb{1} + i\theta\hat{A} + \frac{(i\theta\hat{A})^2}{2!} + \frac{(i\theta\hat{A})^3}{3!} + \dots$$

$$= \mathbb{1} + i\theta A - \frac{\theta^2\hat{A}^2}{2!} - \frac{i\theta^3\hat{A}^3}{3!} + \dots$$

b) $$\cos\theta = 1 - \frac{\theta^2}{2!} + \dots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \dots$$

c) Using $\hat{A}^2 = \mathbb{1}$, we can also conclude $\hat{A}^3 = \hat{A}^2 \cdot \hat{A} = \hat{A}$.

Using these,

$$e^{i\theta\hat{A}} = \mathbb{1} + i\theta\hat{A} - \frac{\theta^2}{2!}\mathbb{1} - \frac{i\theta^3\hat{A}}{3!} + \dots$$

$$\overset{(!)}{=} \left(1 - \frac{\theta^2}{2!}\right)\mathbb{1} + i\cancel{A}\left(\theta - \frac{\theta^3}{3!}\right)\hat{A} + \dots$$

To the order considered, we get

$$\cos\theta\,\mathbb{1} + i\sin(\theta)\,\hat{A}$$

(!Note of caution: Rearrangements in <u>infinite</u> sums is <u>not</u> always so trivial ! It is here though, so no problem. You may look this up for fun)

d. In H.W 1 Q8, we seperated the sum for $e^{i\theta}$ into even & odd powers. We do the same here.

$$e^{i\theta\hat{A}} = \sum_{n=0}^{\infty} \frac{(i\theta A)^n}{n!}$$

2n+1 is always odd.

$$= \sum_{n=0}^{\infty} \frac{(i)^{2n}\theta^{2n}\hat{A}^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i)^{2n+1}\theta^{2n+1}\hat{A}^{2n+1}}{(2n+1)!}$$

Since n is a natural number, 2n is always even.

Now note that $(i^{2n}) = (i^2)^n = (-1)^n$.

Also, $\hat{A}^{2n} = (\hat{A}^2)^n = (\mathbb{1})^n = \mathbb{1}$. (So this only works if $\hat{A}^2 = \mathbb{1}$!!)

Similarly, $\hat{A}^{2n+1} = A \cdot \hat{A}^{2n} = \hat{A}$.

In consequence of the foregoing,

$$e^{i\theta\hat{A}} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{2n!}}_{\text{Maclaurin for } \cos\theta} \mathbb{1} + i\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!}}_{\text{Maclaurin for } \sin\theta} \hat{A}$$

$$= \boxed{\cos\theta\,\mathbb{1} + i\sin\theta\,\hat{A}}$$

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Q.7. Claim: $|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$

satisfies $i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$.

We check this now:

$$i\hbar\frac{d}{dt}\left(e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle\right)$$

$$= i\hbar\frac{d}{dt}\left(e^{-\frac{i}{\hbar}\hat{H}t}\right)|\psi(0)\rangle$$ (∵ Since $\frac{d|\psi(0)\rangle}{dt} = 0$ since $|\psi(0)\rangle$ is a constant.)

$$\overset{*}{=} i\cancel{\hbar}\left(\frac{-i}{\cancel{\hbar}}\hat{H}\right)e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$$

$$= \hat{H}\underbrace{e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle}_{|\psi(t)\rangle}$$

$= \hat{H}|\psi(t)\rangle$. which is what we claimed!

(In *, we have treated $\hat{H}$ as a number, not an operator. This was allowed by the question. of course, it is not true. Think about the limit def of $\frac{d}{dt}$ to see how to derivate operators!)

Q. 8.

a) Since $\hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$$\hat{H} = b\hbar \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b\hbar \\ b\hbar & 0 \end{bmatrix}.$$

b) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

$$|\psi(t)\rangle = \frac{e^{\frac{-i}{\hbar}\hat{H}t}}{\sqrt{2}}(|0\rangle + |1\rangle).$$

$$e^{\frac{-i}{\hbar}\hat{H}t} = e^{\frac{-i}{\cancel{\hbar}}(b\cancel{\hbar}\hat{X})t} = e^{-ibt\hat{X}}$$

Since $\hat{X}^2 = \mathbb{1}$, we can say

$$= \cos(-bt)\mathbb{1} + i\sin(-bt)\hat{X}$$

$$= \cos(bt)\mathbb{1} - i\sin(bt)\hat{X}.$$

! $\cos(a) = \cos(-a)$ $\forall a \in \mathbb{R}$.
$\sin(-a) = -\sin a$.

8 contd

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\cos(bt)\mathbb{1} - i\sin(bt)\hat{X}\right)\left(|0\rangle + |1\rangle\right).$$

$$= \frac{1}{\sqrt{2}}\left[\cos(bt)|0\rangle + \cos(bt)\mathbb{1} - i\sin(bt)|1\rangle - i\sin(bt)|0\rangle\right]$$

This is since $\mathbb{1}|\text{any state}\rangle = |\text{any state}\rangle$.

& $\hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so $\hat{X}|0\rangle = |1\rangle$

& $\hat{X}|1\rangle = |0\rangle$.

Simplifying, we get

$$= \frac{1}{\sqrt{2}}\left[\left(\cos(bt) - i\sin(bt)\right)|0\rangle + \left(\cos(bt) - i\sin(bt)\right)|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{-ibt}|0\rangle + e^{-ibt}|1\rangle\right].$$

$$= \frac{e^{-ibt}}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] = e^{-ibt}\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right].$$

$$= e^{-ibt}\left[|\psi(0)\rangle\right].$$

This is a global phase.

These are complex nos of the form $e^{i\phi}$ multiplied with the <u>whole</u> state.

Note that global phases do not affect the probability calculations at all. Let's see why.

Suppose I have the state $|\psi\rangle$ & also the state $e^{i\phi}|\psi\rangle$. I would like to find the prob. of getting output ~~to~~ $|n\rangle$.

So I'll do:

for $|\psi\rangle$: $\quad |\langle n|\psi\rangle|^2$

for $e^{i\phi}|\psi\rangle$: $\quad |\langle n|e^{i\phi}|\psi\rangle|^2$

$$= |e^{i\phi}\langle n|\psi\rangle|^2$$

$$= |e^{i\phi}|^2 |\langle n|\psi\rangle|^2$$

$$= \cancel{\langle n|\phi\rangle} |\langle n|\psi\rangle|^2$$

Since $|e^{i\phi}|^2 = 1$.

Since $|\psi(t)\rangle = e^{-ibt}|\psi(0)\rangle$, no measurement probs. change with time ∇

c) $|\psi(0)\rangle = |0\rangle$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|0\rangle = e^{-\frac{i}{\cancel{\hbar}}(b\cancel{\hbar}\hat{X})t}|0\rangle$$

$$= e^{-ibt\hat{X}}|0\rangle$$

$$= [\cos(bt) - i\sin(bt)\hat{X}]|0\rangle.$$

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This yields

$$\cos(bt)\,|0\rangle - i\sin(bt)\,|1\rangle.$$

This is clearly <u>not</u> $|0\rangle$, so one expect that the probs are changing. Let's see

Prob$(|0\rangle)$ =

$$\left|\langle 0|\left(\cos(bt)|0\rangle - i\sin(bt)|1\rangle\right)\right|^2$$

$$= \left|\cos(bt)\langle 0|0\rangle - i\sin(bt)\langle 0|1\rangle\right|^2$$

since $\langle 0|0\rangle = 1$, $\langle 0|1\rangle = 0$

we have

$$|\cos(bt)|^2 = \cos^2(bt).$$

for $P(|1\rangle)$, we get

$$|-i\sin(bt)|^2 = (-i\sin(bt))(i\sin(bt))$$
$$= \sin^2(bt).$$

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