Problem Set 1

In this sheet,

$$|D\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|A\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Question 1

Consider a three qubit system. To represent states of the three-qubit system, we will use the convention that in the composite state $|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle$, the first ket ($|\alpha\rangle$) is of the first qubit, the second ket ($|\beta\rangle$) is of the second qubit, and the third ket ($|\gamma\rangle$) is of the third qubit.

(a) Suppose the first qubit is in the state $|0\rangle$, the second in $|D\rangle$, and the third in $|A\rangle$. Find the composite state of the system (do the expansion!).

(b) Are any two qubits entangled if our system is in the state $\frac{1}{2}(|0\rangle|0\rangle|0\rangle + |0\rangle|1\rangle|0\rangle - |0\rangle|0\rangle|1\rangle - |0\rangle|1\rangle|1\rangle)$?

(c) Repeat the previous part with the state $\frac{1}{\sqrt{2}}(|0\rangle|1\rangle|0\rangle + |1\rangle|0\rangle|0\rangle)$.

(d) Show that the states in parts b,c are normalised.

(e) Suppose our three qubit system is in the state given in part (b). We now do a measurement on our three qubit system. Find the probability that the output state is $|A\rangle|D\rangle|A\rangle$.

(f) Repeat the previous part with our system in the state given in part (c).

Question 2

Consider a four qubit system. To represent states of the four-qubit system, we will use the convention that in the composite state $|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\tau\rangle$, the first ket ($|\alpha\rangle$) is of the first qubit, the second ket ($|\beta\rangle$) is of the second qubit, the third ket ($|\gamma\rangle$) is of the third qubit, and $|\tau\rangle$ is of the fourth qubit.

(a) Suppose the first qubit is in the state $|0\rangle$, the second in $|D\rangle$, the third in $|A\rangle$, and the fourth qubit in $|D\rangle$. Find the composite state of the system (do the expansion!).

(b) Are any two qubits entangled if our system is in the state $\frac{1}{\sqrt{2}}(|0\rangle|1\rangle|0\rangle|0\rangle + |0\rangle|1\rangle|0\rangle|1\rangle)$?

(c) Repeat the previous part with the state $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle|1\rangle|0\rangle - |0\rangle|0\rangle|0\rangle|1\rangle)$.

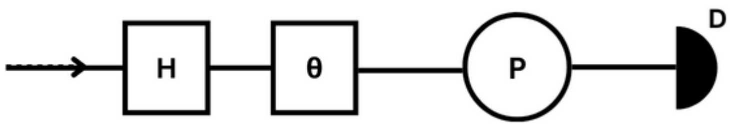
(d) Repeat the previous part with the state $\frac{1}{\sqrt{2}}(|1\rangle|0\rangle|0\rangle|0\rangle - |0\rangle|0\rangle|0\rangle|1\rangle)$. (Hint: you can change the ordering convention we are using to write the four qubit state. Use this to compare the state in this part to the one in the previous part.)

(e) Suppose our four qubit system is in the state given in part (b). We now do a measurement on this system. Find the probability that the output state is $|A\rangle |D\rangle |A\rangle |D\rangle$.

(f) Repeat the previous part with our system in the state given in part (c).

Question 3

Consider the one qubit circuit shown below,



Quantum Circuit for Question 3

where θ represents a phase gate. Its action is stated below.

$$|H\rangle \xrightarrow{\theta} |H\rangle$$
$$|V\rangle \xrightarrow{\theta} e^{i\theta} |V\rangle$$

where θ is a real number.

The action of the H gate is stated below.

$$|H\rangle \xrightarrow{\mathrm{H}} \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$$
$$|V\rangle \xrightarrow{\mathrm{H}} \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle).$$

P represents a polariser with its axis set at ϕ degrees. This means the polariser fully lets through photons found in the state $|P\rangle$. $|P\rangle$ is found by rotating $|V\rangle$ by ϕ degrees anticlockwise.

(a) Find the matrix representing the combined action of the H and phase gate.

(b) If we input the state $|V\rangle$ into this circuit, find the probability (in terms of θ and ϕ) that the detector, D, clicks. The detector clicks if it detects $|H\rangle$.

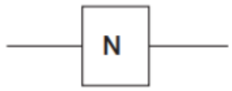
Question 4

Come up with a three-qubit quantum circuit that starts takes the state $|0\rangle |0\rangle |0\rangle$ and converts it into $\frac{1}{\sqrt{2}}(|0\rangle |0\rangle |0\rangle + |1\rangle |1\rangle |1\rangle)$.

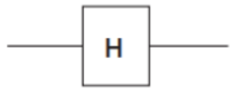
You are allowed to use one or more of the following quantum gates. The action of each is described below.

Note that your circuit will use three lines representing the three qubits.

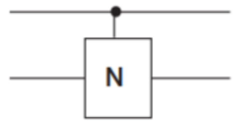
Problem Set 1



Input	Output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$



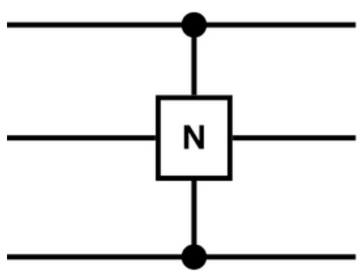
Input	Output
$\|0\rangle$	$\frac{1}{\sqrt{2}}\left(\|1\rangle + \|0\rangle\right)$
$\|1\rangle$	$\frac{1}{\sqrt{2}}\left(\|1\rangle - \|0\rangle\right)$



Input	Output
$\|0\rangle\|0\rangle$	$\|0\rangle\|0\rangle$
$\|0\rangle\|1\rangle$	$\|0\rangle\|1\rangle$
$\|1\rangle\|0\rangle$	$\|1\rangle\|1\rangle$
$\|1\rangle\|1\rangle$	$\|1\rangle\|0\rangle$

Question 5

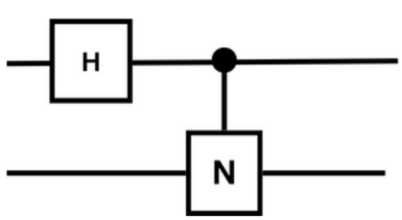
The gate shown flips the second qubit if **both** the first and last qubit are in the state $|1\rangle$. Write the matrix for this gate.



Gate for Question 5

Question 6

If the input state for the following circuit is $|0\rangle|1\rangle$, find the output state. Now, draw a



Quantum Circuit for Question 6

quantum circuit that reverses the operation of this gate. Show that your suggested circuit recovers $|0\rangle|1\rangle$ from the output state you found above.

Question 7

Express the Bell state

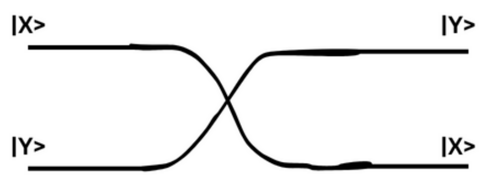
$$\frac{1}{\sqrt{2}}\big(\,|HH\rangle + |VV\rangle\,\big)$$

in terms of the vectors

$$|L\rangle = \frac{1}{\sqrt{2}}\big(\,|H\rangle + i\,|V\rangle\,\big)$$
$$|R\rangle = \frac{1}{\sqrt{2}}\big(\,|H\rangle - i\,|V\rangle\,\big).$$

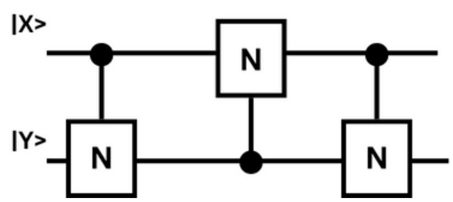
Question 8

The swap gate simply swaps the states of the qubits : inputting $|x\rangle$ on the first qubit and $|y\rangle$ on the second qubit and passing through a swap gate results in $|y\rangle$ on the first qubit and $|x\rangle$ on the second.



Swap Gate

Show that the following sequence of Controlled-Not gates is an interesting way to implement the swap gate.



Quantum Circuit for Question 8

Try this in two ways.

(a) Write down each C-Not gate as a matrix and multiply in the proper order. Compare this with the Swap Gate matrix.

(b) Consider the action of the Swap Gate on the 4 basis states. Compare this with the action of the sequence on C-Not gates on the same. Remember that- since these gates are **linear**- if two circuits do the same thing to all basis states, they do the same thing to **any** input state.

Question 9

Express $|11\rangle$ as a superposition of any two entangled states (Bell states) for the two qubits. Doing this exercise will help you understand that the superposition of entangled states is not necessarily entangled.

Question 10

Consider the Bell state:

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\bigg(|01\rangle - |10\rangle\bigg).$$

(a) Express the entangled state in $|\alpha\rangle$ and $\left|\alpha^\perp\right\rangle$ basis defined as:

$$|\alpha\rangle = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle\,,$$
$$\left|\alpha^\perp\right\rangle = -\sin\alpha\,|0\rangle + \cos\alpha\,|1\rangle\,.$$

(b) Is the state still entangled in the new basis?

(c) Also show that their results will always, always disagree, i.e. if one qubit is projected onto $|\alpha\rangle$, the other will always be projected onto $\left|\alpha^\perp\right\rangle$. It does not matter what α is.

Question 11

The matrix for a phase gate with $\phi = \pi$ is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

What is the matrix for a **square-root of the phase** gate which is a gate that produces a π-gate when applied twice?