Quiz 2

Name : ______________________________

Roll Number: ______________________________

Question 1

In this question, a simple version of the entanglement-swapping protocol is presented. We work with a system of four qubits input to our system shown in Figure 1. The first qubit belongs to Alice, the second and third qubits belong to Bob, and the fourth qubit belongs to Charlie. The first and second qubits are in a Bell state. Furthermore, the third and fourth qubits are also in a Bell state. The Bell state is given by:

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}\Big(\,|00\rangle + |11\rangle\,\Big).$$

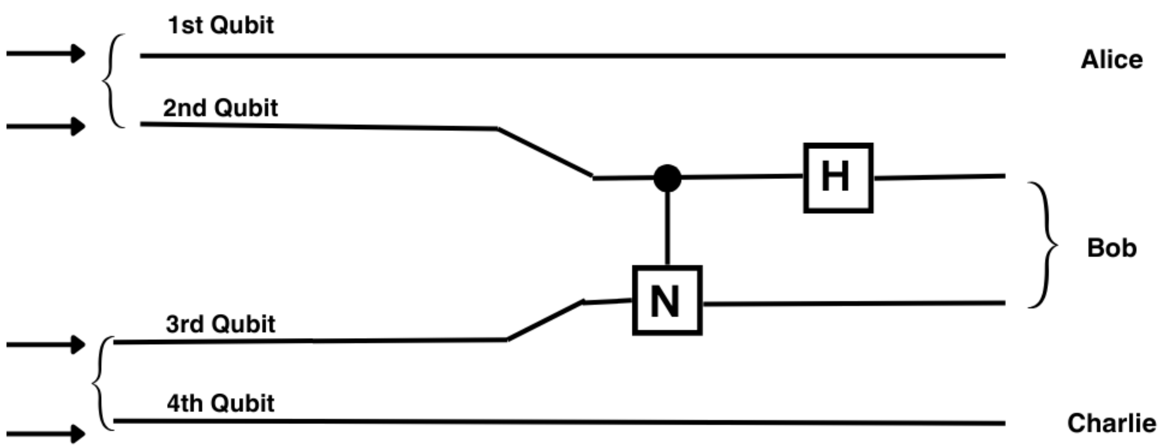

Figure 1: Quantum Circuit for Question 1

Bob applies a set of gates to the qubits he receives. These are defined as:

(i) Controlled NOT gate (controlled N gate): This acts on two qubits. If the control qubit is in the state $|1\rangle$, the qubit with the N gate on it is flipped. If the control qubit is in $|0\rangle$, nothing happens to the qubit with the N gate on it. In either case, nothing happens to the control qubit.

(ii) Hadamard Gate (H gate): This is a single qubit gate.

$$|0\rangle \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|1\rangle \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

(a) Write down the state progression of these four qubits. [4]

(b) Show that Alice's and Charlie's qubits are entangled at the end of the circuit. [2.5]

Question 2

Consider the one qubit circuit shown below,

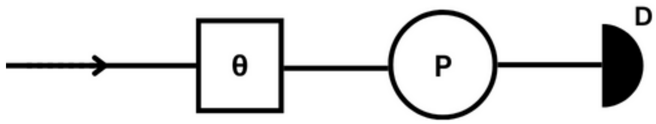

Figure 2: Quantum Circuit for Question 2

where θ represents a phase gate. Its action is stated below.

$$|H\rangle \xrightarrow{\theta} |H\rangle$$
$$|V\rangle \xrightarrow{\theta} e^{i\theta}|V\rangle$$

where θ is a real number.

P represents a polariser with its axis set at ϕ degrees. This means the polariser fully lets through photons found in the state $|P\rangle$. $|P\rangle$ is found by rotating $|H\rangle$ by ϕ degrees anticlockwise.

(a) Find the matrix representing the phase gate. [1.5]

(b) If we input the state $|H\rangle$ into this circuit, find the probability (in terms of θ and ϕ) that the detector, D, clicks. You must make some (valid) effort to compute the mod squared to get full credit for this part. [6]

1 Solution 1

(a)

$$
\begin{aligned}
|\Psi_{in}\rangle &= |\Phi^+\rangle \otimes |\Phi^+\rangle \\
&= \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \otimes \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \\
&= \frac{1}{2}\Big(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle \Big) \\
&\xrightarrow{U_{CNot-23}} \\
&= \frac{1}{2}\Big(|0000\rangle + |0011\rangle + |1110\rangle + |1101\rangle \Big) \\
&\xrightarrow{U_{H-2}} \\
&= \frac{1}{2}\left[|0\rangle \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) |00\rangle + |0\rangle \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) |11\rangle + |1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) |10\rangle + |1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) |01\rangle \right] \\
&= \frac{1}{2\sqrt{2}}\Big[|0000\rangle + |0100\rangle + |0011\rangle + |0111\rangle + |1010\rangle - |1110\rangle + |1001\rangle - |1101\rangle \Big]
\end{aligned}
$$

(b) Let's write Alice's, Bob's and Charlie's qubits separately:

$$
\frac{1}{2\sqrt{2}}\Big[|0\rangle_A |00\rangle_B |0\rangle_C + |1\rangle_A |00\rangle_B |1\rangle_C + |0\rangle_A |01\rangle_B |1\rangle_C + |1\rangle_A |01\rangle_B |0\rangle_C + |0\rangle_A |10\rangle_B |0\rangle_C
$$
$$
- |1\rangle_A |10\rangle_B |1\rangle_C + |0\rangle_A |11\rangle_B |1\rangle_C - |1\rangle_A |11\rangle_B |0\rangle_C \Big].
$$

In each pair, second and third qubits can be factored out, and the first and fourth can be written together to see their entanglement. Factoring out Bob's qubits, we can write the above state as:

$$
\begin{aligned}
\frac{1}{2}\Big[& |00\rangle \otimes \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
& |01\rangle \otimes \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\
& |10\rangle \otimes \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
& |11\rangle \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} \Big].
\end{aligned}
$$

It is clear from the above expression that Alice's and Charlie's qubits are entangled, and can be written as the four Bell states.

Solution 2

(a) $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$.

(b) The state that goes into the circuit is $|H\rangle$. Nothing happens to this state as it passes through the phase gate.

If the photon emerges from P, it must be in the state $|P\rangle = \cos(\phi)\,|H\rangle + \sin(\phi)\,|V\rangle$. To find $|P\rangle$, use this figure below.

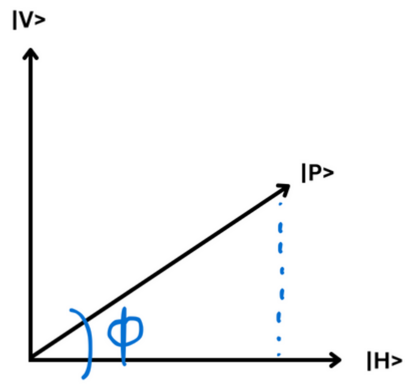

Figure 3: $|P\rangle$

The probability of it surviving through P is

$$|\langle P|H\rangle|^2 = |(\cos(\phi)\,\langle H| + \sin(\phi)\,\langle V|)\,|H\rangle|^2 = \cos^2(\phi).$$

We have used the fact that

$$\langle V|H\rangle = \langle H|V\rangle = 0,$$

and

$$\langle H|H\rangle = \langle V|V\rangle = 1.$$