Quiz 2

Name : ____

Roll Number: ____

Question 1

(a) The GHZ state is an entangled quantum state for 3 qubits and its state is :

$$|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle \Big).$$

Find the probability that **each of the three qubits** are in the diagonal state, where the diagonal state is defined as:

$$|D\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle \Big).$$

[3]

(b) Given the state:

$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\Big(|H\rangle\,|V\rangle - |V\rangle\,|H\rangle \Big).$$

Find the probability that the first qubit is in the $|L\rangle$ state and the second qubit is in the $|R\rangle$ state, where

$$|L\rangle = \frac{1}{\sqrt{2}}\Big(|H\rangle + i\,|V\rangle \Big)$$
$$|R\rangle = \frac{1}{\sqrt{2}}\Big(|H\rangle - i\,|V\rangle \Big).$$

[3]

Question 2

Consider the three-qubit quantum circuit shown overleaf.

The circuit consists of a Hadamard gate (H gate), a SWAP gate, and a controlled-NOT (controlled N gate). The action of each gate is described below.

(i) Hadamard Gate (H gate): This is a single qubit gate.

$$|0\rangle \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|1\rangle \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

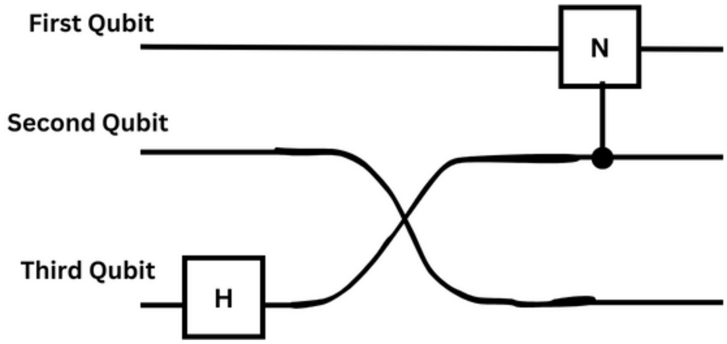

Figure 1: Quantum Circuit for Question 2

(ii) SWAP gate: This is a two-qubit gate. If the input state is $|X\rangle \otimes |Y\rangle$, the output is $|Y\rangle \otimes |X\rangle$. This gate is represented by the wires crossing over in the circuit shown.

(iii) Controlled NOT gate (controlled N gate): This acts on two qubits. If the control qubit is in the state $|1\rangle$, the qubit with the N gate on it is flipped. If the control qubit is in $|0\rangle$, nothing happens to the qubit with the N gate on it. In either case, nothing happens to the control qubit.

To represent states of the three-qubit system, we will use the convention that in the composite state $|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle$, the first ket ($|\alpha\rangle$) is of the first qubit, the second ket ($|\beta\rangle$) is of the second qubit, and the third ket ($|\gamma\rangle$) is of the third qubit.

(a) Write the matrix representing the two-qubit SWAP gate. [3.5]

(b) The input state into the circuit is $|0\rangle |0\rangle |0\rangle$. Find the state at the end of the circuit.

(c) Is the state you found in part (b) entangled? [1.5]

(d) We do a measurement on the final three-qubit state. Find the probability that the third qubit is found in the state $|0\rangle$ and the first two qubits are found in orthogonal basis states (as in, if the first qubit is found in $|0\rangle$, the second should be in $|1\rangle$ or vice versa). [2]

Hint: Getting either $|010\rangle$ or $|100\rangle$ as the output will satisfy the condition set in this part.

Solution 1

(a) We need to find the overlap with the state:

$$
\begin{aligned}
|D\rangle \otimes |D\rangle \otimes |D\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \\
&= \frac{1}{2\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}
\end{aligned}
$$

Furthermore,

$$
|\Psi_{GHZ}\rangle == \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}
$$

We can find the probability as follows:

$$
\begin{aligned}
&|\langle DDD|\Psi_{GHZ}\rangle|^2 \\
&= \left|\frac{1}{2\sqrt{2}}\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}\right|^2 \\
&= \frac{1}{4}.
\end{aligned}
$$

(b) We need to projection on

$$|L\rangle \otimes |R\rangle = \frac{1}{2}\Big(|H\rangle + i\,|V\rangle \Big) \otimes \Big(|H\rangle - i\,|V\rangle \Big)$$
$$= \frac{1}{2}\Big(|HH\rangle - i\,|HV\rangle + i\,|VH\rangle + |VV\rangle \Big)$$

$$Prob. = \Big| \langle L|\,\langle R|\,(\big|\Psi^-\big\rangle) \Big|^2$$
$$= \left(\frac{1}{2\sqrt{2}}\right)^2 \Big|(i+i)\Big|^2$$
$$= \frac{1}{2}.$$

Solution 2

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $|000\rangle \xrightarrow{H_3} |00\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|000\rangle + |001\rangle).$

$\frac{1}{\sqrt{2}}(|000\rangle + |001\rangle) \xrightarrow{\text{SWAP}_{2,3}} \frac{1}{\sqrt{2}}(|000\rangle + |010\rangle).$

$\frac{1}{\sqrt{2}}(|000\rangle + |010\rangle) \xrightarrow{\text{CNOT}_{2,1}} \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |0\rangle = |\psi_{\text{out}}\rangle\,.$

(c) Partially, yes. The first and second qubits are entangled with each other, while being separable from the third.

(d) $\text{P}(|010\rangle) = |\,\langle 010|\psi_{\text{out}}\rangle\,|^2 = \frac{1}{2}|\,\langle 010|000\rangle + \langle 010|110\rangle\,|^2 = 0.$

$\text{P}(|100\rangle) = |\,\langle 100|\psi_{\text{out}}\rangle\,|^2 = \frac{1}{2}|\,\langle 100|000\rangle + \langle 100|110\rangle\,|^2 = 0.$

$\text{P}(|010\rangle) + \text{P}(|100\rangle) = 0.$