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Assignment 2. Solution

Q1] Since $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ & $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

* $a|0\rangle + ib|1\rangle = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + ib\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ ib \end{bmatrix}$

(ii) so $ib|0\rangle + a|1\rangle = \begin{bmatrix} ib \\ a \end{bmatrix}$

(iii) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & (iv) $\begin{bmatrix} 0 \\ a \end{bmatrix}$

1(b) Since $\langle 1| = [0, 1]$ & $\langle 0| = (1, 0)$

$a(1, 0) - ib(0, 1)$

(i) $(a, -ib)$

(ii) $(-ib, a)$

(iii) $(1, 0)$

(iv) $(0, a)$

(c) $\langle \psi_1 | \psi_1 \rangle = (a, ib)\begin{pmatrix} a \\ -ib \end{pmatrix} = a^2 \cancel{+} - i^2 b^2$

$= a^2 + b^2.$

$\langle \psi_2 | \psi_4 \rangle = (-ib, a)\begin{pmatrix} 0 \\ a \end{pmatrix} = 0 + a^2$

$= a^2.$

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(iii) $\langle \psi_1 | \psi_2 \rangle$

$$(a, -ib) \cdot \begin{bmatrix} ib \\ a \end{bmatrix} = i\cancel{a}b - i\cancel{a}b = 0$$

(d) $|\psi_a\rangle = \sqrt{\frac{1}{2}}\, a\,|0\rangle + \sqrt{\frac{1}{2}}\, ib\,|1\rangle$

$\langle \psi_b | = \sqrt{\frac{1}{3}} \langle 0| + \sqrt{\frac{2}{3}}\, i \langle 1|$

$\langle \psi_a | = ?$, $\sqrt{\frac{1}{2}}\,a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{1}{2}}\, ib \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so $\langle \psi_a | = \underline{\sqrt{\frac{1}{2}}}\, a\,(1, 0) - \underline{\sqrt{\frac{1}{2}}}\, ib\,(0\ \ 1)$

$\langle \psi_a | = \left(\sqrt{\frac{1}{2}}\,a,\ 0\right) + \left(0,\ \sqrt{\frac{1}{2}}\,ib\right)$

$\left(\sqrt{\frac{1}{2}}\,a\right)$ = prb amplitude for $\langle 0|$

$\left(-\sqrt{\frac{1}{2}}\,ib\right)$ = " " for $\langle 1|$

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$\langle \psi_b |$ given

* $|\psi_b\rangle$ should have $\sqrt{\frac{1}{3}}$ for $|0\rangle$ & $-\sqrt{\frac{2}{3}}i$ for $|1\rangle$

$\left(\sqrt{\frac{1}{3}}\right)$ = prb amp for $|0\rangle$

& $\left(-\sqrt{\frac{2}{3}}i\right)$ = prb amp for $|1\rangle$.

$\langle \psi_b | \psi_a \rangle$

$\langle \psi_b | = \sqrt{1/3}\,(1,0) + \sqrt{2/3}\,i\,(0,1)$

$= \left(\sqrt{\frac{1}{3}} \quad \sqrt{\frac{2}{3}}i\right)$

$|\psi_a\rangle = \begin{pmatrix} \sqrt{1/2}\,a \\ \sqrt{1/2}\,ib \end{pmatrix}$

$\langle \psi_b | \psi_a \rangle = \left(\sqrt{\frac{1}{3}} \quad \sqrt{2/3}\,i\right) \begin{pmatrix} \sqrt{1/2}\,a \\ \sqrt{1/2}\,ib \end{pmatrix}$

$= \left(\sqrt{1/6}\,a + \sqrt{\frac{2}{6}}\,i^2 b\right) = \sqrt{1/6}\,a - \sqrt{\frac{2}{6}}\,b$

$= \sqrt{\frac{1}{6}}\,a - \sqrt{\frac{1}{3}}\,b$

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(ii) $\langle \psi_a | \psi_b \rangle$

$$\langle \psi_a | = \left(\sqrt{\tfrac{1}{2}}\, a \; , \; -\sqrt{1/2}\, ib \right)$$

$$| \psi_b \rangle = \begin{pmatrix} \sqrt{\tfrac{1}{3}} \\ -\sqrt{2/3}\, i \end{pmatrix}$$

$$\langle \psi_a | \psi_b \rangle = \left(\sqrt{\tfrac{1}{2}}\, a \; , \; -\sqrt{\tfrac{1}{2}}\, ib \right) \begin{pmatrix} \sqrt{1/3} \\ -\sqrt{2/3}\, i \end{pmatrix}$$

$$= \sqrt{\frac{1}{6}}\, a + \sqrt{\frac{2}{6}}\, i^2 b$$

$$= \sqrt{\frac{1}{6}}\, a - \sqrt{\frac{1}{3}}\, b .$$

Question 2

(a) $|\psi\rangle = a|0\rangle + b|1\rangle$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

(b) Complex conjugate $\langle\psi| = a^*\langle 0| + b^*\langle 1|$

$= \begin{pmatrix} a^* & b^* \end{pmatrix}$

(c) Transformation $\rightarrow \begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$

(d) $\hat{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$\hat{R}|0\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$

Similary

$\hat{R}|1\rangle = -\sin\theta|0\rangle + \cos\theta|1\rangle$

(e) $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ ib \end{pmatrix}$

(f) $\hat{R}|0\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$

$= \underbrace{\cancel{\cos(\tfrac{\pi}{2})|0\rangle}}_{0} + \sin(\tfrac{\pi}{2})|1\rangle$

$= |1\rangle$

(g) $$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\theta = \frac{\pi}{2}$$

$$= \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

(2) $$\cancel{\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -a \\ -ib \end{pmatrix}}$$

~~Let's check reverse order.~~

$$\cancel{\begin{pmatrix} a & ib \\ ib & \end{pmatrix}}$$

(2) $$\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -a \\ -ib \end{pmatrix}$$

Let's check reverse order

(1) $$\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} ib \\ a \end{pmatrix}$$

(2) $$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} ib \\ a \end{pmatrix} = \begin{pmatrix} ib\cos\theta - a\sin\theta \\ ib\sin\theta + a\cos\theta \end{pmatrix}$$

Therefore, if we change the order in which transformation are performed, we do not get the same output!

Question 5

(a) State progression

Input state $|1\rangle \otimes |H\rangle$

$$|1\rangle \otimes |A\rangle \xrightarrow{B_1} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |A\rangle$$

$$|A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$$ * Doing the tensor product

$$\rightarrow \frac{1}{2}\left(|0H\rangle - \cancel{|0V\rangle} - \cancel{|1H\rangle} + |1V\rangle\right)$$

- Normalized State after Polarizers

$$\frac{1}{\sqrt{2}}\left(|0H\rangle + |1V\rangle\right)$$

$$\xrightarrow{\text{Mirrors}} \frac{1}{\sqrt{2}}\left(|1H\rangle + |0V\rangle\right)$$

$$\xrightarrow{B_2} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes H + \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes V\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0H\rangle - |1H\rangle) + (|0V\rangle + |1V\rangle)\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}}|0\rangle \otimes (|H\rangle + |V\rangle) - \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)\right\}$$

$$\frac{1}{\sqrt{2}}\left((|0\rangle \otimes |D\rangle) - |1\rangle \otimes |A\rangle \right)$$

H,V polarizers $\cancel{\frac{1}{\sqrt{2}}\left(|0H\rangle + |1V\rangle \right)}$

Now the probability that D_1 clicks is given by

$$= \left| \langle 0| \otimes \langle H| \left(\frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle) - (|1\rangle \otimes |A\rangle) \frac{1}{\sqrt{2}} \right) \right|^2$$

$$= \cancel{\langle 0H| \left\{ \frac{1}{\sqrt{2}} \right(}$$

$$\left| \langle 0H| \left\{ \frac{1}{2} |0H\rangle + |0V\rangle \right\} \right|^2$$

$$= \frac{1}{4}$$

However, the over all probability is $\frac{1}{2} \times \frac{1}{4}$ because $\frac{1}{2}$ of the photons were absorbed by the initial polarizers

(c) In the simple interferometer, if the input state is $|1\rangle$, we would expect D_2 to click with a 100% probability. However, due to the given polarizers, both detectors click. ~~Th~~ The polarizers act as tags by giving us which path information. Hence, there is uncertainty about the momentum of the photons.

(d) Over all the state progression stays the same before it sees the ~~detectors~~ polarizers placed before the detectors.

The final probabilities will change.

$$P(D_1) = \left| \langle 0| \otimes \langle A| \left\{ \frac{1}{\sqrt{2}} |0\rangle \otimes |D\rangle - |1\rangle \otimes |A\rangle \right\} \right|^2$$

$$= 0$$

$$P(D_2) = \left| \langle 1| \otimes \langle A| \left\{ \frac{1}{\sqrt{2}} |0\rangle \otimes |D\rangle - |1\rangle \otimes |A\rangle \right\} \right|^2$$

$$= \frac{1}{2}$$

Over all probability $= 1/4$

(e) The horizontal and vertical polarizers placed after B1 give which path information by tagging different paths with different polarization. This in turn influences the interference behaviour at the second beam splitter. As a result, no interferene happens and both detectors click with equal probability.

However, if we place anti-diagnol polarizers after second beam splitter, we are removing the which path information. This then recoups interference like behavior. The ~~removal of~~ "erasing" of which path information is known as quantum erasure.

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Q4) $|0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}.$$

$$\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}.$$

operations in vector notation!

* Input → B_1 → Mirrors → B_2 → Detector!

$$\begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{B_1} \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix} \xrightarrow{M} \begin{pmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} \xrightarrow{B_2} \begin{pmatrix}0\\-1\end{pmatrix}.$$

* intuitively we can fill in the progression path & then figure out the matrix operations to fill in.

* One may also do it directly starting w/ vectors & figuring it out along the way!

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B splitter $\Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Mirror $\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{B_1} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ +1 & -1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

<u>Mirror</u>

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

<u>B_2</u> $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} + \frac{1}{2} \\ -\frac{1}{2} - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Prb along $D_1 = 0$

Prb " $D_2 = 1$

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(c) Now

Input $\rightarrow B_1 \rightarrow$ Mirrors $\rightarrow$ Device $\rightarrow B_2 \rightarrow$ Detector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{B_1} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

Mirror $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$

Device $|1\rangle \rightarrow i|1\rangle$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ adds a factor (i) to "vertical" input.

Device

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}.$$

B_2 $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} = \begin{pmatrix} -1/2 + i/2 \\ -\frac{1}{2} - i/2 \end{pmatrix}.$

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* Now probability along ~~[illegible]~~ $|0\rangle$ & $|1\rangle$ is possible as evident from the ~~[illegible]~~ final state!

$$\frac{1}{2}\begin{pmatrix} 1+i \\ -1-i \end{pmatrix}$$

This basically means.

$$\frac{1}{2}(1+i)\,|0\rangle \quad \& \quad \frac{-1}{2}(1+i)\,|1\rangle.$$

$$\text{Prb } |0\rangle \Rightarrow \frac{1}{4}(1+i)(1-i) = \frac{1}{4}(1-i^2)$$

$$\text{Prb } |1\rangle \Rightarrow \frac{1}{4}(1+i)(1-i) = \frac{1}{4}(1-i^2)$$

$$= \frac{1}{4}(1-(-1)) = \frac{2}{4} \Rightarrow \frac{1}{2}.$$

Probabilities are halved for each outcome!

Question 3

(a) $|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$. Note that we could have arrived at this by doing the following.

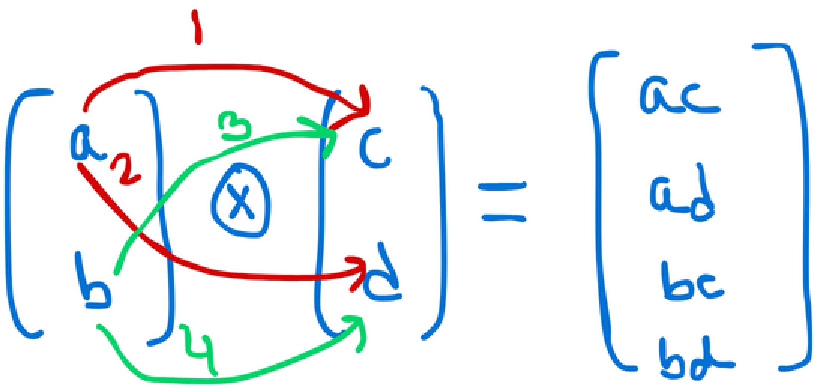

Figure 1: A method to carry out the direct product

(b) $\begin{bmatrix} (ac)^*, (ad)^*, (bc)^*, (bd)^* \end{bmatrix} \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix} = |ac|^2 + |ad|^2 + |bc|^2 + |bd|^2$. You can get the same by doing the following. Note that while taking the inner product for composite states, we compute the inner product within each subsystem and then multiply the results. For example, here note that $|\psi\rangle$ comes from qubit A and $|\phi\rangle$ from qubit B.

$$(\langle\psi| \otimes \langle\phi|)(|\psi\rangle \otimes |\phi\rangle) = \langle\psi|\psi\rangle \langle\phi|\phi\rangle$$

(c) One way to proceed is to figure out the output state for each of the four basis states $|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle$. But since we know the matrix for each of the two gates, we can also perform the tensor product as shown below. Here the first matrix represents the first qubit's gate. The second matrix, denoted by B_2, represents the second qubit's gate. Note that, in this case, both are 2x2 matrices. The tensor product yields a 4x4 matrix. This larger matrix can be constructed out of 4 2x2 matrices as shown.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes B_2 = \left[\begin{array}{c|c} aB_2 & bB_2 \\ \hline cB_2 & dB_2 \end{array}\right]$$

Figure 2: A method to carry out the direct product

In our particular example, this works out as shown below.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & i & i & -1 \\ i & 1 & -1 & i \\ i & -1 & 1 & i \\ -1 & i & i & 1 \end{bmatrix}$$

Note that before we can write our kets down as column vectors, we have to pick a convention for which basis is represented by which slot. For 2 dimensional systems, we have been using the convention shown below.

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}.$$

If we want matrix multiplication to work, this convention also dictates how we write down the matrix for operators. Suppose some operator O acts as defined below.

$$|0\rangle \xrightarrow{O} v\,|0\rangle + w\,|1\rangle$$
$$|1\rangle \xrightarrow{O} x\,|0\rangle + y\,|1\rangle\,.$$

Its matrix in this basis is shown below.

$$\begin{bmatrix} v & x \\ w & y \end{bmatrix}.$$

The first column is the output vector when $|0\rangle$ is the input. The positions of v, w are dictated by our convention for column vectors. Similarly, the first column is the output for $|0\rangle$ and the second column is the output for $|1\rangle$ since our column vectors talk about $|0\rangle$ first and then $|1\rangle$. This reasoning extends to the 4-dimensional examples done above.

(d) The gate acts as stated below.

$$\begin{aligned}
|0\rangle \otimes |0\rangle &\to |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |0\rangle|1\rangle) \\
|0\rangle \otimes |1\rangle &\to |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |0\rangle|1\rangle) \\
|1\rangle \otimes |0\rangle &\to |1\rangle \otimes |0\rangle \\
|1\rangle \otimes |1\rangle &\to |1\rangle \otimes |1\rangle
\end{aligned}$$

The matrix then becomes

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(e) The input state is $|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |0\rangle|1\rangle)$. The corresponding column vector is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The output state can now be found.

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Question 6

(a) We will use the convention that

$$|H\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |V\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So our matrix becomes $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

(b) $\begin{bmatrix} \cos(\phi_1) & -\sin(\phi_1) \\ \sin(\phi_1) & \cos(\phi_1) \end{bmatrix}\begin{bmatrix} \cos(\phi_0) & -\sin(\phi_0) \\ \sin(\phi_0) & \cos(\phi_0) \end{bmatrix} = \begin{bmatrix} \cos(\phi_0+\phi_1) & -\sin(\phi_0+\phi_1) \\ \sin(\phi_0+\phi_1) & \cos(\phi_0+\phi_1) \end{bmatrix}$.

Note that the order of multiplication does not matter in this particular case.

To derive the matrix above, we used the following trigonometric identities.

$$\sin(\phi_0+\phi_1) = \sin(\phi_0)\cos(\phi_1) + \cos(\phi_0)\sin(\phi_1),$$

$$\cos(\phi_0+\phi_1) = \cos(\phi_0)\cos(\phi_1) - \sin(\phi_0)\sin(\phi_1).$$

(c) $\begin{bmatrix} \cos(2\phi) & -\sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{bmatrix}$

(d) Suppose N=3. We can combine the matrix in part c with the matrix in part a using our result in part b to get $\begin{bmatrix} \cos(3\phi) & -\sin(3\phi) \\ \sin(3\phi) & \cos(3\phi) \end{bmatrix}$. We can repeat this exercise then for N=4 and so on.

A way to formalize the foregoing is to suppose that our claim works for N-1 gates. So for N-1 gates, our matrix is $\begin{bmatrix} \cos((N-1)\phi) & -\sin((N-1)\phi) \\ \sin((N-1)\phi) & \cos((N-1)\phi) \end{bmatrix}$. Using our result in part b, we can easily show that the matrix for N gates is $\begin{bmatrix} \cos(N\phi) & -\sin(N\phi) \\ \sin(N\phi) & \cos(N\phi) \end{bmatrix}$. So if we know our result works for N=2 gates, this discussion mandates it is also true for N=3. Then the N=3 result mandates that the result hold for N=4 and so on. This method is known as induction.

(e) The combined action of the N gates can be represented by one gate with rotation angle $= N(\frac{\pi}{2N}) = \frac{\pi}{2}$. So $|H\rangle$ is transformed into $|V\rangle$. Since our detector detects $|H\rangle$, it never clicks: the probability of clicking is zero.

(f) After the first rotation gate, our photon is in the state $\cos\left(\frac{\pi}{2N}\right)|H\rangle + \sin\left(\frac{\pi}{2N}\right)|V\rangle$. The photon allows $|H\rangle$ to pass through. This happens with the probability

$$|\langle H|(\cos\left(\frac{\pi}{2N}\right)|H\rangle + \sin\left(\frac{\pi}{2N}\right)|V\rangle)|^2 = |\cos\left(\frac{\pi}{2N}\right)\langle H|H\rangle + \sin\left(\frac{\pi}{2N}\right)\langle H|V\rangle|^2 = \cos^2\left(\frac{\pi}{2N}\right).$$

The state right before entering the second rotation gate is $|H\rangle$.

(g) Since there are N such instances and we would like the state to make it through every time, the probability of $|H\rangle$ surviving (and hence the detector clicking) is

$$\cos^2\left(\frac{\pi}{2N}\right)\cos^2\left(\frac{\pi}{2N}\right)\cdots\cos^2\left(\frac{\pi}{2N}\right) = \cos^{2N}\left(\frac{\pi}{2N}\right).$$

(h) $\cos^{2N}(\theta) \approx 1 - 2N\theta^2$. Plugging in $\theta = \frac{\pi}{2N}$ yields $1 - \frac{\pi^2}{2N}$. As $N \to \infty$, the second term dies out and our probability approaches 1.

Question 7

(a) The combined quantum space with three qubits is 8-dimensional. The basis states are now $|0\rangle|0\rangle|0\rangle, |0\rangle|0\rangle|1\rangle, |0\rangle|1\rangle|0\rangle, |0\rangle|1\rangle|1\rangle, |1\rangle|0\rangle|0\rangle, |1\rangle|0\rangle|1\rangle, |1\rangle|1\rangle|0\rangle, |1\rangle|1\rangle|1\rangle$.

Only the last two states are affected by the Toffoli gate.

$$|1\rangle\,|1\rangle\,|0\rangle \xrightarrow{\text{Toffoli}} |1\rangle\,|1\rangle\,|1\rangle$$
$$|1\rangle\,|1\rangle\,|1\rangle \xrightarrow{\text{Toffoli}} |1\rangle\,|1\rangle\,|0\rangle\,.$$

Using the convention that

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

our matrix for this gate becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Question 8

(a) The Hadamard gate acts on the first qubit to give the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|0\rangle)$. This state then passes through the controlled-NOT gate to give $\frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle)$.

(b) The last gate the state passed through is the controlled-NOT gate. Let's try to reverse that first. By a "reverse" gate, we mean a gate that can start with the output state of the controlled-NOT gate and map it back to its input.

The action of the controlled-NOT gate is defined below.

$$\begin{aligned}
|0\rangle\,|0\rangle &\xrightarrow{\text{controlled-NOT}} |0\rangle\,|0\rangle \\
|0\rangle\,|1\rangle &\xrightarrow{\text{controlled-NOT}} |0\rangle\,|1\rangle \\
|1\rangle\,|0\rangle &\xrightarrow{\text{controlled-NOT}} |1\rangle\,|1\rangle \\
|1\rangle\,|1\rangle &\xrightarrow{\text{controlled-NOT}} |1\rangle\,|0\rangle\,.
\end{aligned}$$

We can reverse this action by flipping the state of the second qubit if the first qubit is in the state $|1\rangle$. But this is just the controlled-NOT gate: to reverse this controlled-NOT gate, we use another controlled-NOT gate!

The next step (the order of the gates matters!!) is to reverse the Hadamard gate which acts on the first qubit. You can verify (or guess based on our experience with balanced Mach-Zehnder Interferometers) that a Hadamard gate can be reversed by another Hadamard gate. We can fill in the box now.

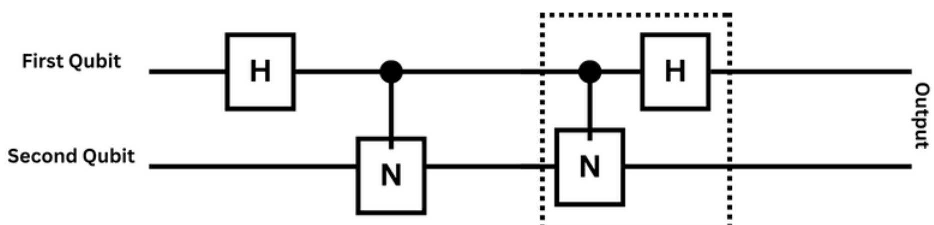

Figure 3: Completed circuit for Question 8.b.

We can also look at this more abstractly. Let's call our initial state $|\psi\rangle$. The matrix for the H gate is represented by U_H. The state after passing through it is then $U_H\,|\psi\rangle$. The matrix for the controlled-NOT is represented by U_N. The state after passing through it is $U_N U_H\,|\psi\rangle$. To get back to $|\psi\rangle$, we first undo U_N with its inverse U_N^{-1}. This gets us $U_N^{-1}U_N U_H\,|\psi\rangle = U_H\,|\psi\rangle$. We then reverse U_H with U_H^{-1} to get $U_H^{-1}U_N^{-1}U_N U_H\,|\psi\rangle = U_H^{-1}U_H\,|\psi\rangle = |\psi\rangle$. Using the preceding discussion about how these reversal gates behave, you should be able to make a truth table for them, and then find the matrices for U_H^{-1} and U_N^{-1}. By comparing these with U_H and U_N respectively, you should be able to, once again, conclude that $U_H^{-1} = U_H$ and $U_N^{-1} = U_N$.

Additional (and more technical note):

There is another way to find the matrices U_H^{-1} and U_N^{-1}. Let's first note that both U_H and U_N are **unitary**. This means that they do not change the "size" of any vector they act on. By "size" of some vector $|\psi\rangle$, we mean $\langle\psi|\psi\rangle$. Let's verify my claim. U_N acts on $|\psi_{in}\rangle$ to give $|\psi_{out}\rangle = U_N |\psi_{in}\rangle$. The corresponding bra vector is given by $\langle\psi_{out}| = \langle\psi_{in}| U_N^\dagger$. $U_N^\dagger$ is the **adjoint** of U_N. So the size of the output vector is $\langle\psi_{out}|\psi_{out}\rangle = \langle\psi_{in}| U_N^\dagger U_N |\psi_{in}\rangle$. If $U_N^\dagger U_N = \mathbb{1}$, $\langle\psi_{out}|\psi_{out}\rangle = \langle\psi_{in}|\psi_{in}\rangle$. So to check if a given matrix is unitary, we check that $U_N^\dagger U_N = \mathbb{1}$. Check this for U_N and U_H. To do so, you will need the matrix for their adjoints. The method to find that is spelled out below.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}^T = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$$

Once you conclude that both are unitary, you immediately know their inverses! This is since $U_N^\dagger U_N = \mathbb{1}$ implies that $U_N^{-1} = U_N^\dagger$. Ditto for U_H.

In the preceding discussion, $\mathbb{1}$ is the **identity matrix**. The special thing about $\mathbb{1}$ is that $\mathbb{1}|\psi\rangle = |\psi\rangle$ for any $|\psi\rangle$. The 2-dimensional identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) Our discussion in the preceding part paid no heed to which state is being input. We found gates that reverse the entire action of the gates earlier in the circuit. Our circuit then can recoup *any* input state.