Problem Set 1 - Solution

Q.1. a) $|0\rangle \otimes |D\rangle \otimes |A\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left(|000\rangle - |001\rangle + |010\rangle - |011\rangle\right)$$

$$= \frac{1}{2}\left(|000\rangle - |001\rangle + |010\rangle - |011\rangle\right) = |\psi_a\rangle$$

b) No. Part (a) shows that the state given in (b) is entirely separable.

c) $\frac{1}{\sqrt{2}}(|010\rangle + |100\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes |0\rangle$

we see that the first two qubits are in a Bell state : the first two qubits are entangled.

d) The bra corresponding to the state given in (a) =

$\frac{1}{2}\left(\langle 000| - \langle 001| + \langle 010| - \langle 011|\right) = \langle \psi_a|$

~~P(ψ)~~ $\langle \psi_a|\psi_a\rangle =$

$$\frac{1}{2}\left(\langle 000| - \langle 001| + \langle 010| - \langle 011|\right)\frac{1}{2}\left(|000\rangle - |001\rangle + |010\rangle - |011\rangle\right)$$

$$= \frac{1}{4}\left(\langle 000|000\rangle + \langle 001|001\rangle + \langle 010|010\rangle + \langle 011|011\rangle\right)$$

$$= \frac{1}{4}(4) = 1$$

The cross terms (i.e. terms like $\langle 001|010\rangle$) give zero.

~~To see this: $\langle 001|010\rangle = \langle 0|\langle 0|\langle 1| \otimes |0\rangle|1\rangle|0\rangle$~~

we do a similar calculation with the state in part c.

e) The state in part (b) can be written as

$$|0\rangle \otimes |D\rangle \otimes |A\rangle = |\psi_{in}\rangle$$

we want the ~~input~~ output to be $|A\rangle \otimes |D\rangle \otimes |A\rangle = |\psi_{out}\rangle$.

The corresponding bra vector $= \langle \psi_{out}| = \langle A| \otimes \langle D| \otimes \langle A|$.

we now do $\langle \psi_{out}|\psi_{in}\rangle =$

$$\Big(\underbrace{\langle A|}_{\text{first qubit bra}}\ \underbrace{\langle D|}_{\text{second qubit bra}}\ \underbrace{\langle A|}_{\text{third qubit bra}}\Big)\Big(\underbrace{|0\rangle}_{\text{first qubit ket}}\ \underbrace{|D\rangle}_{\text{second qubit ket}}\ \overbrace{|A\rangle}^{\text{third qubit ket}}\Big)$$

$$= \langle A|0\rangle \langle D|D\rangle \langle A|A\rangle$$ } the bras & kets of the <u>same</u> qubits meet.

$$= \langle A|0\rangle\ \underbrace{(1)\ (1)}_{\text{since } |D\rangle \text{ \& } |A\rangle \text{ are normalised.}}$$

$$\downarrow$$

$$\frac{1}{\sqrt{2}}\big(\langle 0| - \langle 1|\big)|0\rangle$$

$$= \frac{1}{\sqrt{2}}\big(\langle 0|0\rangle - \langle 1|0\rangle\big) = \frac{1}{\sqrt{2}}$$

$$\text{Prob} = |\langle \psi_{out}|\psi_{in}\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

! If you find it difficult to keep track of which qubit belongs to which system, put little labels. I do this in the next part.

f) $|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\left(|010\rangle + |100\rangle\right)$.

I call the first qubit "qubit A", the second one "B", the third one "C". I put these labels on the qubits so I can follow which qubit is which.

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\left(|0_A\rangle|1_B\rangle|0_C\rangle + |1_A\rangle|0_B\rangle|0_C\rangle\right).$$

The desired out $= |0_A\rangle|D_B\rangle|A_C\rangle = |\psi_{out}\rangle$

$$\langle\psi_{out}| = \langle 0_A|\langle D_B|\langle A_C|.$$

$\langle\psi_{out}|\psi_{in}\rangle =$

$$\langle 0_A|\langle D_B|\langle A_C|\left(\frac{1}{\sqrt{2}}\left(|0_A\rangle|1_B\rangle|0_C\rangle + |1_A\rangle|0_B\rangle|0_C\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}}\Big(\langle 0_A|0_A\rangle\langle D_B|1_B\rangle\langle A_C|0_C\rangle + \langle 0_A|1_A\rangle\langle D_B|0_B\rangle\langle A_C|0_C\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}\left(\langle D_B|1_B\rangle\langle A_C|0_C\rangle\right)$$ since $\langle 0_A|1_A\rangle = 0$ & $\langle 0_A|0_A\rangle = \langle 1_A|1_A\rangle = 1$.

$$\langle D_B|1_B\rangle = \frac{1}{\sqrt{2}}\left(\langle 0_B| + \langle 1_B|\right)\left(|1_B\rangle\right) = \frac{1}{\sqrt{2}}.$$

$$\langle A_C|0_C\rangle = \frac{1}{\sqrt{2}}\left(\langle 0_C| + \langle 1_C|\right)\left(|0_C\rangle\right) = \frac{1}{\sqrt{2}}.$$

So $\langle \psi_{out} | \psi_{in} \rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$

$$\text{Prob} = |\langle \psi_{out} | \psi_{in} \rangle|^2 = \left| \frac{1}{2\sqrt{2}} \right|^2 = \frac{1}{4(2)} = \frac{1}{8}.$$

Question 2 (I use qubit labelling throughout this question.)

a)

$$|0_A\rangle \otimes |D_B\rangle \otimes |A_C\rangle \otimes |D_D\rangle$$

$$= |0_A\rangle \otimes \frac{1}{\sqrt{2}}(|0_B\rangle + |1_B\rangle) \otimes \frac{1}{\sqrt{2}}(|0_C\rangle - |1_C\rangle) \otimes \frac{1}{\sqrt{2}}(|0_D\rangle + |1_D\rangle).$$

(I call the fourth qubit "D".)

$$= \frac{1}{2\sqrt{2}}\Big(|0_A0_B0_C0_D\rangle + |0_A0_B0_C1_D\rangle - |0_A0_B1_C0_D\rangle - |0_A0_B1_C1_D\rangle$$
$$+ |0_A1_B0_C0_D\rangle + |0_A1_B0_C1_D\rangle - |0_A1_B1_C0_D\rangle - |0_A1_B1_C1_D\rangle\Big)$$

b) $\frac{1}{\sqrt{2}}\Big(|0_A\rangle|1_B\rangle|0_C\rangle|0_D\rangle + |0_A\rangle|1_B\rangle|0_C\rangle|1_D\rangle\Big)$

$$= |0_A\rangle \otimes |1_B\rangle \otimes |0_C\rangle \otimes \frac{1}{\sqrt{2}}(|0_D\rangle + |1_D\rangle)$$

Since our state is <u>fully</u> seperable, no two qubits are entangled.

c) $\frac{1}{\sqrt{2}}\left(\underline{|0_A\rangle|0_B\rangle}|1_C\rangle|0_D\rangle - \underline{|0_A\rangle|0_B\rangle}|0_C\rangle|1_D\rangle\right)$

$= |0_A\rangle \otimes |0_B\rangle \otimes \frac{1}{\sqrt{2}}\left(|1_C\rangle|0_D\rangle - |0_C\rangle|1_D\rangle\right)$

qubits C & D are in a Bell state.
So those two qubits are entangled.

d) It was easy to see how to start factorisation above since $|0_A\rangle|0_B\rangle$ (the first two qubits) were the same in both. ~~So~~

Lets now look at

~~$\frac{1}{\sqrt{2}}\left(|1_A\rangle|0_B\rangle|0_C\rangle|0_D\rangle - |0_A\rangle|0_B\rangle|0_C\rangle|1_D\rangle\right)$~~

$\frac{1}{\sqrt{2}}\left(|1_A\rangle\underline{|0_B\rangle|0_C\rangle}|0_D\rangle - |0_A\rangle\underline{|0_B\rangle|0_C\rangle}|1_D\rangle\right)$

Here $|0_B\rangle|0_C\rangle$ appears in both, but it is ~~of~~ sandwitched between the states of A & D. To check if A & D are entangled, lets try to pull out $|0_B\rangle \otimes |0_C\rangle$.

Note that

$$|\psi\rangle \otimes |\phi\rangle \neq |\phi\rangle \otimes |\psi\rangle$$

generally since the first slot describes qubit A & the other qubit B (by our convention).
But, if we put labels for our qubits, there is nothing wrong. i.e. $|\psi_A\rangle \otimes |\phi_B\rangle = |\phi_B\rangle \otimes |\psi_A\rangle$

This is since in both states ($|\psi_A\rangle \otimes |\phi_B\rangle$ & $|\phi_B\rangle \otimes |\psi_A\rangle$, qubit A $\rightarrow |\psi\rangle$
& qubit B $\rightarrow |\phi\rangle$).

Now I can do:

$$\frac{1}{\sqrt{2}}\left(|0_B\rangle|0_C\rangle|1_A\rangle|0_D\rangle - |0_B\rangle|0_C\rangle|0_A\rangle|1_D\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0_B\rangle \otimes |0_C\rangle \otimes \left(|1_A\rangle|0_D\rangle - |0_A\rangle|1_D\rangle\right)\right).$$

$$= |0_B\rangle \otimes |0_C\rangle \otimes \frac{1}{\sqrt{2}}\left(|1_A\rangle|0_D\rangle - |0_A\rangle|1_D\rangle\right).$$

Clearly, qubits A, D occupy a Bell state & so are <u>entangled</u>.

e) State in part b = $|0_A\rangle \otimes |1_B\rangle \otimes |0_C\rangle \otimes |D_D\rangle$

desired output = $|A_A\rangle \otimes |D_B\rangle \otimes |A_C\rangle \otimes |D_D\rangle$.

$\langle \text{output} | \text{input} \rangle$

$= \cancel{\langle 0_A | A_A}\ \langle A_A | 0_A\rangle \langle D_B | 1_B\rangle \langle A_C | 0_C\rangle \langle D_D | D_D\rangle.$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)(1) = \frac{1}{2\sqrt{2}}$$

$$\text{prob} = \left|\langle \text{output} | \text{input} \rangle\right|^2 = \left|\frac{1}{2\sqrt{2}}\right|^2$$

$$= \frac{1}{(2)^2(\sqrt{2})^2} = \frac{1}{8}.$$

f)

State in part c $= |0_A\rangle \otimes |0_B\rangle \otimes \frac{1}{\sqrt{2}}\left(|1_C\rangle|0_D\rangle - |0_C\rangle|1_D\rangle\right)$

desired output $= |A_A\rangle \otimes |D_B\rangle \otimes |A_C\rangle \otimes |D_D\rangle$.

$\cancel{\langle \text{output} | \text{input} \rangle = \langle A_A | 0_A \rangle \langle D_B | 0_B \rangle}$

~~desired~~ $\langle \text{output} | \text{input} \rangle =$

$$\langle A_A | 0_A \rangle \langle D_B | 0_B \rangle \left(\langle A_C | \otimes \langle D_D|\right)\left(\frac{1}{\sqrt{2}}\left(|1_C\rangle|0_D\rangle - |0_C\rangle|1_D\rangle\right)\right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\langle A_C | 1_C \rangle \langle D_D | 0_D \rangle - \langle A_C | 0_C \rangle \langle D_D | 1_D \rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{-1}{2\sqrt{2}}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{-1}{2\sqrt{2}}$$

$$\text{prob} = \left|\langle \text{out} | \text{in} \rangle\right|^2 = \left|\frac{-1}{2\sqrt{2}}\right|^2$$

$$= \frac{1}{8}.$$

3. a) Let's figure out what H & phase gate do to our basis, $\{|H\rangle, |V\rangle\}$. We can use this to write the matrix.

$$|H\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \xrightarrow{\theta} \frac{1}{\sqrt{2}}(|H\rangle + e^{i\theta}|V\rangle)$$

$$|V\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \xrightarrow{\theta} \frac{1}{\sqrt{2}}(|H\rangle - e^{i\theta}|V\rangle).$$

Our matrix:

$$\begin{array}{cc} & \begin{array}{cc} |H\rangle & |V\rangle \end{array} \leftarrow \text{inputs} \\ \begin{array}{c} |H\rangle \\ |V\rangle \\ \text{outputs} \end{array} & \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}e^{i\theta} & -\frac{1}{\sqrt{2}}e^{i\theta} \end{pmatrix} \end{array} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ e^{i\theta} & -e^{i\theta} \end{pmatrix}$$

You can find the same by multiplying the matrices in order.

Since H acts first, first we write its matrix: U_H.

$$U_H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then the phase gate, U_P.

$$U_P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

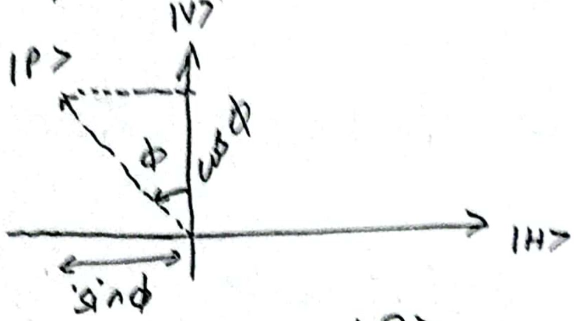
Our combined matrix is given by:

$$U_P U_H.$$

b. $|V\rangle \to \frac{1}{\sqrt{2}}\left(|H\rangle - e^{i\theta}|V\rangle\right)$.

This state enters the polariser.

let's find $|P\rangle$ first:



$-\sin\phi\,|H\rangle + \cos\phi\,|V\rangle = |P\rangle$.

The probability of our state surviving through the polariser:

$$\left|\langle P|\left(\frac{1}{\sqrt{2}}\left(|H\rangle - e^{i\theta}|V\rangle\right)\right)\right|^2$$

$$= \left|\left(-\sin\phi\,\langle H| + \cos\phi\,\langle V|\right)\frac{1}{\sqrt{2}}\left(|H\rangle - e^{i\theta}|V\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\right|^2 \left|\left(-\sin\phi\,\langle H|H\rangle + \cancel{\sin\phi\, e^{i\theta}\langle H|V\rangle}^{\,0} + \cancel{\cos\phi\,\langle V|H\rangle}^{\,0} - \cos\phi\, e^{i\theta}\langle V|V\rangle\right)\right|^2$$

$$= \frac{1}{2}\left|-\sin\phi - \cos\phi\, e^{i\theta}\right|^2 = \frac{1}{2}\left|\sin\phi + \cos\phi\, e^{i\theta}\right|^2$$

$$= \frac{1}{2}\left(\sin\phi + \cos\phi\, e^{i\theta}\right)\left(\sin\phi + \cos\phi\, e^{-i\theta}\right)$$

$$= \frac{1}{2}\left(\sin^2\phi + \sin\phi\cos\phi\, e^{-i\theta} + \sin\phi\cos\phi\, e^{i\theta} + \cos^2\phi\right)$$

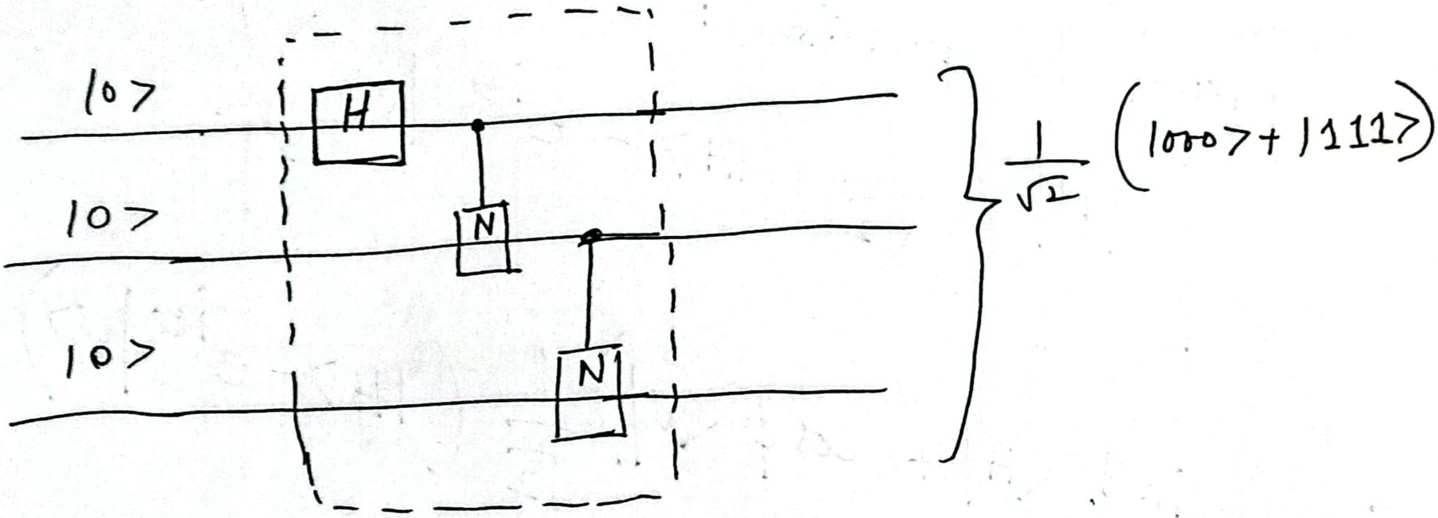
$$= \frac{1}{2}\left(\sin^2\phi + \cos^2\phi + \sin\phi\cos\phi\left(e^{i\theta} + e^{-i\theta}\right)\right)$$

$$= \frac{1}{2}\left(1 + \sin\phi\cos\phi\left(2\cos\theta\right)\right)$$

$$= \frac{1}{2}\left(1 + 2\sin\phi\cos\phi\cos\theta\right)$$

$$= \frac{1}{2}\left(1 + \sin(2\phi)\cos\theta\right).$$

4.



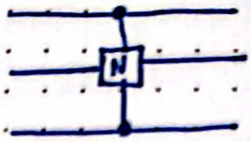
$$|000\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle).$$

$$\xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle)$$

$$\xrightarrow{CNOT_{2,3}} \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

* Question 5

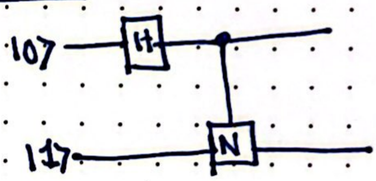


$$
\begin{array}{c|cccccccc}
 & |000\rangle & |010\rangle & |100\rangle & |001\rangle & |110\rangle & |011\rangle & |101\rangle & |111\rangle \\
\hline
|000\rangle & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
|010\rangle & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
|100\rangle & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
|001\rangle & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
|110\rangle & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
|011\rangle & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
|101\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
|111\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

8x8 matrix

* dashed lines drawn to help you keep track of row and column.

Question 6

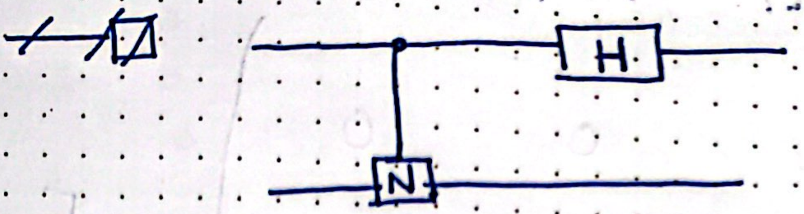


$$|0\rangle \otimes |1\rangle \xrightarrow[\text{gate}]{\text{Hadamard}} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |1\rangle$$

$$= \frac{|01\rangle + |11\rangle}{\sqrt{2}}$$

$$\xrightarrow{\text{Controlled-Not}} \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

Quantum circuit that reverses the operation ⇒



Output $\dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$

Controlled-Not $\longrightarrow \dfrac{|01\rangle + |11\rangle}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{\cancel{2}}$

Hadamard on 1st qubit $\longrightarrow \dfrac{1}{2}\left\{ |01\rangle + \cancel{|11\rangle} + |01\rangle - \cancel{|11\rangle} \right\}$

$= \dfrac{\cancel{2}\,|01\rangle}{\cancel{2}} = |01\rangle$

Question 7

$|L\rangle - |R\rangle = \dfrac{1}{\sqrt{2}}\left(2i|V\rangle\right)$

$= i\sqrt{2}\,|V\rangle$

$|V\rangle = \dfrac{-i}{\sqrt{2}}\left(|L\rangle - |R\rangle\right)$

$|L\rangle + |R\rangle = \dfrac{2}{\sqrt{2}}|H\rangle$

$\dfrac{1}{\sqrt{2}}\left(|L\rangle + |R\rangle\right) = |H\rangle$

$|HH\rangle = \dfrac{1}{2}\left(|LL\rangle + \cancel{|LR\rangle} + \cancel{|RL\rangle} + |RR\rangle\right)$

$|VV\rangle = \dfrac{1}{2}\left(|LL\rangle - \cancel{|LR\rangle} - \cancel{|RL\rangle} + |RR\rangle\right)$

$|HH\rangle + |VV\rangle = |LL\rangle + |RR\rangle$

Therefore $\dfrac{1}{\sqrt{2}}\left(|HH\rangle + |VV\rangle\right) = \dfrac{1}{\sqrt{2}}\left(|LL\rangle + |RR\rangle\right)$

Question 8

$$
\begin{array}{c|cccc}
 & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\
\hline
|00\rangle & & & & \\
|01\rangle & & & & \\
|10\rangle & & & & \\
|11\rangle & & & &
\end{array}
\quad
\begin{array}{c}
 \\ |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

First gate

$$
\begin{array}{c}
 \\ |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle
\end{array}
\begin{array}{c}
\begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\end{array}
$$

Second gate

The tird gate is the same as first gate.

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

* This is the same matrix as swap gate!

8b) Our goal is to show that the circuit shown behaves exactly like the SWAP gate. We want to do this without any matrix algebra. Here's how: let's first ask how we if two circuits are the same. They are the same if they do the same thing to <u>any</u> input state.

What's the most general input state here? Well, we have <u>2</u> qubits so

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$a, b, c, d \in \mathbb{C}$.

Let the swap gate be rep. by U_s.

$$U_s|\psi\rangle = \cancel{a\, U_s|\psi\rangle}$$

$$a(U_s|00\rangle) + b(U_s|01\rangle) + c(U_s|10\rangle) + d(U_s|11\rangle).$$

This is what we mean when we say U_s acts "<u>linearly</u>". (i.e. $U_s(k|\phi\rangle + m|\psi\rangle) = k(U_s|\phi\rangle) + m(U_s|\psi\rangle)$).

So ~~if~~ ~~there~~ what matters is only what U_s does to the 4 basis states! If we can design some circuit that has the matrix U_c such that U_c & U_s do the same thing to all basis states, we would have shown that that the SWAP gate & this circuit do the same thing to any input (so they have the same action).

So let's compare the action of SWAP on our basis:

$$|00\rangle \xrightarrow{SWAP} |00\rangle$$
$$|01\rangle \longrightarrow |10\rangle$$
$$|10\rangle \longrightarrow \cancel{|01\rangle}\ |01\rangle$$
$$|11\rangle \longrightarrow |11\rangle.$$

Now, run $|00\rangle$ through the circuit:

$$|00\rangle \xrightarrow{C\text{-}NOT} |00\rangle \xrightarrow{CNOT} |00\rangle \xrightarrow{CNOT} |00\rangle.$$

Now, $|01\rangle$

$$|01\rangle \xrightarrow{CNOT} |11\rangle \xrightarrow{CNOT} |10\rangle \xrightarrow{CNOT} |10\rangle.$$

Now, $|10\rangle$

$$|10\rangle \xrightarrow{CNOT} |11\rangle \xrightarrow{CNOT} |01\rangle \xrightarrow{CNOT} |01\rangle.$$

Now, $|11\rangle$

$$|11\rangle \xrightarrow{CNOT} |10\rangle \longrightarrow |10\rangle \xrightarrow{CNOT} |11\rangle.$$

Since the action matches on the basis, it matches for all inputs.

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Q9]

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]$$

$$|11\rangle = \frac{1}{\sqrt{2}}\left[|\phi^+\rangle - |\phi^-\rangle\right]$$

$$= \left(\frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] - \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]\right)\frac{1}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{2}}|11\rangle\right)$$

$$= \frac{2}{2}|11\rangle = |11\rangle.$$

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Q10) $|\psi\rangle = \frac{1}{\sqrt{2}}\left[\,|01\rangle - |10\rangle\,\right]$

$$\begin{pmatrix} |\alpha\rangle \\ |\alpha^{\perp}\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}}_{A} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}$$

$$\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = A^{-1} \begin{pmatrix} |\alpha\rangle \\ |\alpha^{\perp}\rangle \end{pmatrix}$$

$$A^{-1} = \frac{1}{\text{Adj}A} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$\text{Adj}A = \cos^2\alpha - (-\sin^2\alpha) = \cos^2\alpha + \sin^2\alpha$

$= 1.$

$$A^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} |\alpha\rangle \\ |\alpha^{\perp}\rangle \end{pmatrix}$$

NO:

DATE:

$$|0\rangle = A^{-1}_{11}|\alpha\rangle + A^{-1}_{12}|\alpha^{\perp}\rangle \rightarrow (1)$$

$$|1\rangle = A^{-1}_{21}|\alpha\rangle + A^{-1}_{22}|\alpha^{\perp}\rangle \rightarrow (2)$$

* $A^{-1}_{11} = \cos\alpha$ & $A^{-1}_{12} = -\sin\alpha$

* $A^{-1}_{21} = \sin\alpha$ & $A^{-1}_{22} = \cos\alpha$

$$|0\rangle = \cos\alpha|\alpha\rangle + [-\sin\alpha]|\alpha^{\perp}\rangle$$
$$|1\rangle = \sin\alpha|\alpha\rangle + \cos\alpha|\alpha^{\perp}\rangle$$

Since

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right]$$

NO:

DATE:

$$|0\rangle = \cos\alpha\,|\alpha\rangle - \sin\alpha\,|\alpha^{\perp}\rangle$$
$$|1\rangle = \sin\alpha\,|\alpha\rangle + \cos\alpha\,|\alpha^{\perp}\rangle .$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right).$$

$$|01\rangle = \cos\alpha\sin\alpha\,|\alpha\alpha\rangle + \cos^2\alpha\,|\alpha\rangle|\alpha^{\perp}\rangle$$
$$- \sin^2\alpha\,|\alpha^{\perp}\rangle|\alpha\rangle - \sin\alpha\cos\alpha\,\cancel{|\alpha^{\perp}\alpha^{\perp}\rangle}\ ta$$

$$|10\rangle = \cancel{\sin\alpha\cos\alpha\,|\alpha\alpha\rangle} - \sin^2\alpha\,|\alpha\alpha^{\perp}\rangle$$
$$+ \cos^2\alpha\,|\alpha^{\perp}\alpha\rangle - \cos\alpha\sin\alpha\,\cancel{|\alpha^{\perp}\alpha^{\perp}\rangle}$$

$$|01\rangle - |10\rangle = \cos^2\alpha\,|\alpha\alpha^{\perp}\rangle - \sin^2\alpha\,|\alpha^{\perp}\alpha\rangle$$
$$- \cos^2\alpha\,|\alpha^{\perp}\alpha\rangle + \sin^2\alpha\,|\alpha\alpha^{\perp}\rangle$$

using $\cos^2\theta + \sin^2\theta = 1$

$$= |\alpha\alpha^{\perp}\rangle - |\alpha^{\perp}\alpha\rangle$$

states are still entagled!

$$\langle \alpha \alpha | \left[\langle \alpha \alpha^{+} \rangle - \langle \alpha^{+} \alpha \rangle \right]$$

$= 0$] Same result for $\langle \alpha^{+} \alpha^{+} |$

NO:
DATE:

Q11) $\Phi = \pi$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$P(\Phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\Phi i} \end{pmatrix}$$ if $\Phi = \pi$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{\Phi i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2}{2}\Phi i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Phi/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Phi/2} \end{pmatrix}$$

so matrix $= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Phi/2} \end{pmatrix}$

for $\Phi = \pi$ $= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$