Question 1

The time-dependent Schrödinger equation can be used to determine the evolution of a quantum state after some time t. It is written as:

$$i\hbar\frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(0)\rangle\,,$$

where $\hat{H}$ is the Hamiltonian of the system. In this question, we define $\hat{H}$ to be:

$$\hat{H} = b\hbar\hat{Y}.$$

(a) Show that $\hat{Y}^2 = \mathbb{1}$.

(b) A quantum state $|\psi(0)\rangle$ at time $t = 0$ is given by

$$\frac{1}{\sqrt{2}}\Big(\,|0\rangle + |1\rangle\,\Big).$$

What is $|\psi(t)\rangle$ if $bt = \frac{\pi}{2}$?

(c) Identify $|\psi(0)\rangle$ and $|\psi(t)\rangle$ on the Bloch sphere.

(d) If $bt = \frac{\pi}{2}$, what rotation will take $|\psi(0)\rangle$ to $|\psi(t)\rangle$? Specify the amount and axis of rotation.

(e) Now suppose that the initial state is

$$\frac{1}{\sqrt{2}}\Big(\,|0\rangle + i\,|1\rangle\,\Big).$$

How does this state evolve with time?

(f) For part (f), is there any measurement that can differentiate between $|\psi(0)\rangle$ and the state at some later time?

The three Pauli operators are defined below

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Question 2

We define an operator $\hat{U} = e^{-i\theta\frac{\hat{Y}}{2}}$ where $\theta = \frac{\pi}{2N}$. Let $N = 10$.

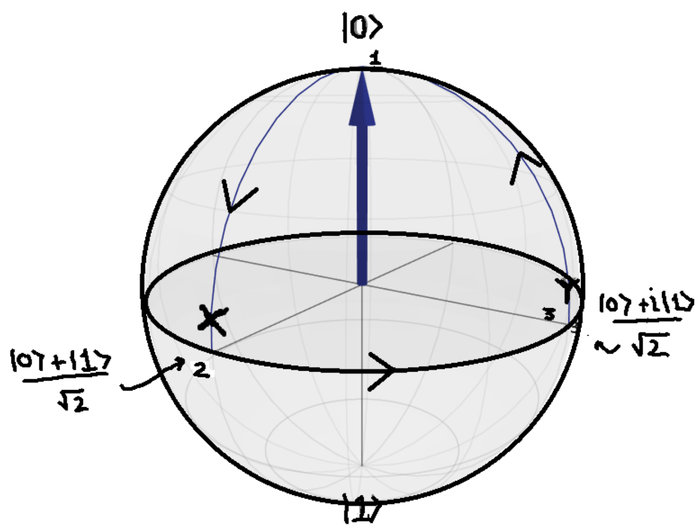
(a) Our initial state is $|0\rangle$. We act $\hat{U}$ on this state N times, in succession. Find the final state.

(b) Our initial state is again $|0\rangle$. We act $\hat{U}$ on it and then perform a measurement along the $|0\rangle$ direction. Performing a "measurement along the $|0\rangle$ direction" on the state $a\,|0\rangle + b\,|1\rangle$ yields $|0\rangle$ with probability $|a|^2$.

We carry out this process — acting $\hat{U}$ and then measuring along the $|0\rangle$ direction — N times. What is the probability that we have $|0\rangle$ at the end of the process?

Question 3

Write down **one** matrix that rotates the state $|0\rangle$ from **1** to **2** to **3** and back to **1**. This trajectory is shown on the Bloch sphere.



Quiz 3
(Variant 2)

(a) $\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\hat{Y}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\mathbb{1}}$$

$\hat{H} = b\hbar\hat{Y}$

(b) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle = e^{-ibt\hat{Y}}|\psi(0)\rangle$$

$\rightarrow$ Given $bt = \frac{\pi}{2}$ $\quad \hat{U} = e^{-i\frac{\pi}{2}\hat{Y}} = \cos\left(\frac{\pi}{2}\right)\hat{\mathbb{1}} - i\sin\left(\frac{\pi}{2}\right)\hat{Y}$

$= -i\hat{Y}$

$$|\psi(t)\rangle = -i\hat{Y}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\hat{Y}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{Y}|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle \qquad \hat{Y}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

(c) A general state on the Bloch Sphere is given by

$$\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$$

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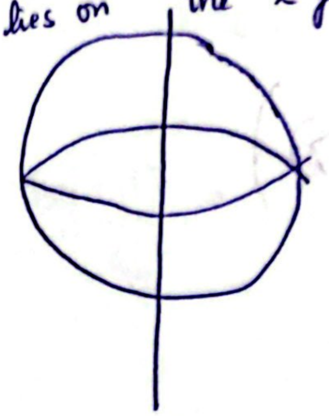
$\theta = \frac{\pi}{2}$ for $\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}$ and $e^{i\phi} = -1$

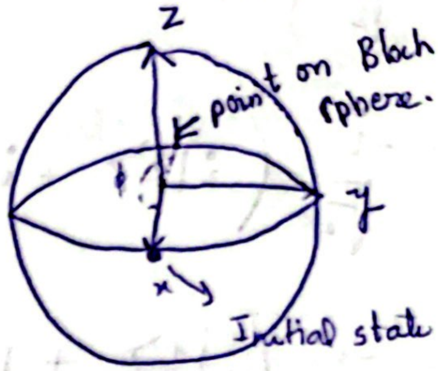
$\phi = \pi$

Using $\theta = \frac{\pi}{2}$ and $\phi = \pi$

and ignoring the global phase

our state lies on the x-y plane.





(d) Axis $\rightarrow$ Y anticlockwise.

Angle of rotation is π.

(e) Given the unitary time evolution operator

$-i\hat{Y}$

$$|\psi(t)\rangle = -i\hat{Y}\,|\psi(0)\rangle = i\hat{Y}\,\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

$$= \cancel{\frac{1}{\sqrt{2}}\left(-|1\rangle + i|0\rangle\right)}$$

$$\cancel{= \frac{-1}{\sqrt{2}}\left(\right.}$$

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$\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$

$= \frac{-i}{\sqrt{2}}\left(\hat{Y}|0\rangle + i\hat{Y}|1\rangle\right)$

$= \frac{-i}{\sqrt{2}}\left(i|1\rangle + i(-i|0\rangle)\right)$

$= \frac{-i}{\sqrt{2}}\left(i|1\rangle + |0\rangle\right)$

$= \frac{-i}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \leftarrow$ a global phase of i picked up!

The state remains the same.

(f) There is no meaurement that can tell apart the two states since only a global phase is picked up.

Question 2

(a) $\hat{U} = e^{\frac{-i\theta\hat{Y}}{2}}$ where $\theta = \frac{\pi}{2N} = \frac{\pi}{20}$.

$\hat{U} = e^{-i\frac{\pi}{40}\hat{Y}}$

$(\hat{U})^{10} = e^{-i\frac{\pi}{40}\hat{Y}}$

Let's first do $e^{-i\frac{\pi}{40}\hat{Y}} \cdot e^{-i\frac{\pi}{40}\hat{Y}}$ ~~...~~

$= e^{-i\frac{\pi}{20}(\hat{Y}+\hat{Y})}$

$e^{-i\frac{\pi}{40}\hat{Y}} = e^{-i\frac{\pi}{40}(Y+Y+Y+Y\ldots)}$

$= e^{-i\frac{\pi}{4}\hat{Y}}$

$e^{-i\frac{\pi}{4}\hat{Y}} = \left(\frac{1}{\sqrt{2}}\hat{I} - \frac{i}{\sqrt{2}}\hat{Y}\right)|0\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}(i|1\rangle)$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

(b) $e^{-i\frac{\pi}{40}\hat{Y}} = \left\{\cos\left(\frac{\pi}{40}\right)\hat{I} - i\sin\left(\frac{\pi}{40}\right)\hat{Y}\right\}|0\rangle$

$= \cos\left(\frac{\pi}{40}\right)|0\rangle - i\sin\left(\frac{\pi}{40}\right)(i|1\rangle)$

$= \cos\left(\frac{\pi}{40}\right)|0\rangle + \sin\left(\frac{\pi}{40}\right)|1\rangle$

Measurement $\rightarrow$ $P(0) = \left|\langle 0|\left\{\cos\left(\frac{\pi}{40}\right)|0\rangle + \sin\left(\frac{\pi}{40}\right)|1\rangle\right\}\right|^2$

$= \cos^2\left(\frac{\pi}{40}\right)$ ← Probability after 1 ~~measurement~~ measurement.

→ State collapses to $|0\rangle$ and we apply Unitary operator again. We do this 10 times and the final state is again $|0\rangle$.

The final state is $|0\rangle$ with probability.

$$\cos^2\left(\frac{\pi}{40}\right) \cdot \cos^2\left(\frac{\pi}{40}\right) \cdots\cdots \cos^2\left(\frac{\pi}{40}\right)$$

$$\left[\cos^2\left(\frac{\pi}{40}\right)\right]^{10} = \cos^{20}\left(\frac{\pi}{40}\right).$$

Question 3

1-2 $\quad |0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $\quad \mathcal{U}_y\left(\frac{\pi}{2}\right)$ will achieve this state.

Unitary operator for the rotation $\Rightarrow$ $e^{-i\frac{\theta}{2}\hat{Y}} = e^{-i\frac{\pi}{4}\hat{Y}}$

$$e^{-i\frac{\pi}{4}\hat{Y}}|0\rangle = \left(\cos\left(\frac{\theta}{4}\right)\hat{\mathbb{1}} - i\sin\left(\frac{\theta}{4}\right)\hat{Y}\right)|0\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}(i|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

2→3 $\quad \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $\quad \mathcal{U}_z\left(\frac{\pi}{2}\right) = e^{-i\frac{\pi}{4}\hat{Z}}$

$$e^{-i\frac{\pi}{4}\hat{Z}}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \cancel{\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}\hat{Z}}\left(\frac{1}{\sqrt{2}}\hat{\mathbb{1}} - \frac{i}{\sqrt{2}}\hat{Z}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(|0\rangle + |1\rangle - i|0\rangle + i|1\rangle\right)$$

$$= \left(\frac{1-i}{\sqrt{2}}\right)\frac{|0\rangle}{\sqrt{2}} + \frac{|1\rangle}{\sqrt{2}}\left(\frac{1+i}{\sqrt{2}}\right)$$

$$= \left(\frac{1-i}{\sqrt{2}}\right)\frac{|0\rangle}{\sqrt{2}} + \frac{i|1\rangle}{\sqrt{2}}\left(\frac{-i+1}{\sqrt{2}}\right)$$

$$= \left(\frac{1-i}{\sqrt{2}}\right)\left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}\right)$$

We achieve the gate with a phase factor of $e^{-\frac{i\pi}{4}}$.

• 3 → 1 $\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \rightarrow |0\rangle$ This is quite simple

$$\downarrow$$
$$\mathcal{U}_x\left(\frac{\pi}{2}\right)$$

$$\mathcal{U}_x\left(\frac{\pi}{2}\right) = e^{-i\frac{\pi}{4}\hat{X}} = \left(\frac{1}{\sqrt{2}}\hat{\mathbb{1}} - \frac{i\hat{X}}{\sqrt{2}}\right)$$

$$\mathcal{U}_x\left(\frac{\pi}{2}\right)\left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\hat{\mathbb{1}} - i\frac{\hat{X}}{\sqrt{2}}\right)\left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(|0\rangle + i|1\rangle - i|1\rangle + |0\rangle\right)$$

$$= |0\rangle.$$

Now let's write the matrix.

$$\hat{\mathcal{U}}_y\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\hat{\mathbb{1}} - \frac{i}{\sqrt{2}}\hat{Y} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & +1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\hat{\mathcal{U}}_z\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\hat{\mathbb{1}} - \frac{i}{\sqrt{2}}\hat{Z} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}$$

$$\hat{\mathcal{U}}_x(\pi/2) = \frac{1}{\sqrt{2}}\hat{\mathbb{1}} - \frac{i}{\sqrt{2}}\hat{X} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$\hat{\mathcal{U}}_x\hat{\mathcal{U}}_z\hat{\mathcal{U}}_y = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \leftarrow \text{Ans}$$

Simplify.