Question 1

The time-dependent Schrödinger equation can be used to determine the evolution of a quantum state after some time t. It is written as:

$$i\hbar \frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(0)\rangle\,,$$

where $\hat{H}$ is the Hamiltonian of the system. In this question, we define $\hat{H}$ to be:

$$\hat{H} = b\hbar\hat{X}.$$

(a) Show that $\hat{X}^2 = \mathbb{1}$.

(b) A quantum state $|\psi(0)\rangle$ at time $t = 0$ is given by

$$\frac{1}{\sqrt{2}}\Big(|0\rangle + i\,|1\rangle \Big).$$

What is $|\psi(t)\rangle$ if $bt = \frac{\pi}{2}$?

(c) Identify $|\psi(0)\rangle$ and $|\psi(t)\rangle$ on the Bloch sphere.

(d) If $bt = \frac{\pi}{2}$, what rotation will take $|\psi(0)\rangle$ to $|\psi(t)\rangle$? Specify the amount and axis of rotation.

(e) Now suppose that the initial state is

$$\frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle \Big).$$

How does this state evolve with time?

(f) For part (f), is there any measurement that can differentiate between $|\psi(0)\rangle$ and the state at some later time?

The three Pauli operators are defined below

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

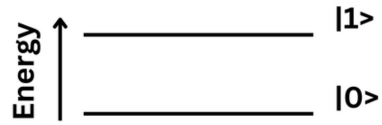


Figure 1: Energy Level Diagram for Question 2

Question 2

If an electron is in the lower energy state, it is labeled as $|0\rangle$, and if it is in the higher (excited) state, it is labeled as $|1\rangle$. A pulse is applied that has the Hamiltonian

$$\hbar V_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Our electron starts in the lower energy state. The pulse switches on at time $t = 0$ and ends at $\frac{T}{V_0}$.
What is the probability that the electron is found in the excited state at the end of the pulse?

Question 3

Show that a π rotation about the $z-$axis followed by a $\frac{\pi}{2}$ rotation about the $y-$axis generates a Hadamard gate.

Quiz 3 Group 1

Q1.

a) $\hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \hat{X}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \hat{\mathbb{1}}$.

since acting $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on any $\begin{bmatrix} a \\ b \end{bmatrix}$ give back $\begin{bmatrix} a \\ b \end{bmatrix}$.

b) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$.

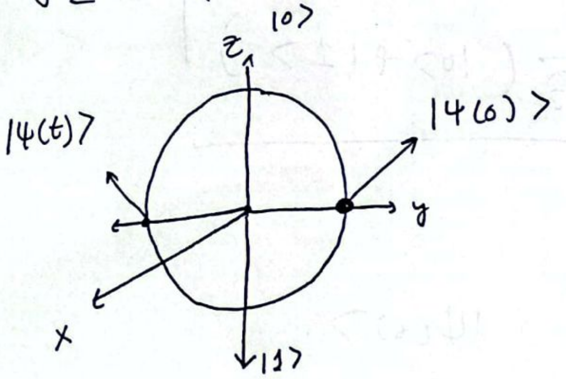
$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle = \frac{e^{-ibt\hat{X}}}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$.

Now $e^{-ibt\hat{X}} = \cos(bt)\,\mathbb{1} - i\sin(bt)\,\hat{X}$

with $bt = \frac{\pi}{2}$, we are left with $-i\hat{X}$ only. Acting this on $|\psi(0)\rangle$ yields

$$\frac{1}{\sqrt{2}}\left(-i|1\rangle + |0\rangle\right) = \boxed{\frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)}$$

c)



$\underline{|\psi(0)\rangle}$

$\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$

$\underline{|\psi(t)\rangle}$

$\theta = \frac{\pi}{2}$, $\phi = \frac{3\pi}{2}$

d) (π). Angle $\to \pi$

X. Axis $\to X$. (anticlockwise).

e) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

$$|\psi(t)\rangle = \left(\cos(bt)\,\mathbb{1} - i\sin(bt)\,\hat{X}\right)\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right].$$

$$= \frac{1}{\sqrt{2}}\left[\left(\cos(bt) - i\sin(bt)\right)|0\rangle + \left(\cos(bt) - i\sin(bt)\right)|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}e^{-ibt}\left(|0\rangle + |1\rangle\right) = \frac{e^{-ibt}}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |\psi(t)\rangle.$$

f) No. Suppose we measure the probability of getting some state as the outcome, ~~to~~ $|n\rangle$.

for $|\psi(0)\rangle$ =

$$\left|\langle n|\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\right|^2$$

for $|\psi(t)\rangle$ =

$$\left|\langle n|\frac{e^{-ibt}}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\right|^2$$

$$= \underbrace{\left|e^{-ibt}\right|^2}_{=1.}\ \underbrace{\left|\langle n|\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\right|^2}_{=\text{ same as for } |\psi(0)\rangle.}$$

Q2.

$$\hat{H} = \hbar V_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hbar V_0 \hat{X}.$$

$$|\psi(0)\rangle = |0\rangle.$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}(\hbar V_0 \hat{X})t} |0\rangle$$

$$= e^{-iV_0 t \hat{X}} |0\rangle$$

we are interested at $t = \frac{T}{V_0}$.

$$|\psi(\tfrac{T}{V_0})\rangle = e^{-iT\hat{X}} |0\rangle$$

$$= \left[\cos(T)\mathbb{1} - i\sin(T)\hat{X}\right] |0\rangle$$

$$= \cos(T) - i\sin(T)\,|1\rangle.$$

$\mathrm{Prob}(|1\rangle) =$

$$|\langle 1|\psi(t)\rangle|^2 = |-i\sin(T)|^2$$

$$= (-i\sin(T))(i\sin(T))$$

$$= \underline{\sin^2(T)}.$$

Q3.

$$|0\rangle \xrightarrow{(\pi)_z} e^{-\frac{i\pi}{2}\hat{z}} |0\rangle = \left[\cos\left(\tfrac{\pi}{2}\right)\mathbb{1} - i\sin\left(\tfrac{\pi}{2}\right)\hat{z}\right]|0\rangle$$

$$= -i|0\rangle.$$

$$-i|0\rangle \xrightarrow{(\frac{\pi}{2})_y} e^{-\frac{i\pi}{4}\hat{Y}}(-i|0\rangle) = \frac{-i}{\sqrt{2}}(\mathbb{1} - i\hat{Y})(|0\rangle)$$

$$= \frac{-i}{\sqrt{2}}\left[|0\rangle + |1\rangle\right].$$

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$|1\rangle \xrightarrow{\left(\frac{\pi}{2}\right)z} i|1\rangle$

Same steps as before

but note that $\hat{z}|0\rangle = |0\rangle$ but

$\hat{z}|1\rangle = -|1\rangle$. !!

$$i|1\rangle \xrightarrow{\left(\frac{\pi}{2}\right)y} e^{\frac{-i\pi}{4}\hat{Y}}\, i|1\rangle$$

$$= \cancel{\frac{i}{\sqrt{2}}[|0\rangle]} \; \frac{i}{\sqrt{2}}\left(\mathbb{1} - i\hat{Y}\right)|1\rangle$$

$$= \cancel{\frac{i}{\sqrt{2}}(|0\rangle}$$

$$= \frac{i}{\sqrt{2}}\left(|1\rangle - |0\rangle\right)$$

$$= \frac{-i}{\sqrt{2}}\left(|0\rangle - |1\rangle\right).$$

we could not have said what we say here if the phases for $|0\rangle$, $|1\rangle$ were different !

Since we pick up the same global phase with $|0\rangle$ & $|1\rangle$, ~~so~~ our rotations will perform the Hadamard gate on any arbitrary state. It will give the correct state with a global phase multiplied. But this can be dropped.

+ Note that $-i = e^{i\frac{3\pi}{2}} = e^{-i\frac{\pi}{2}}$.

+ Also any arbitrary state is $a|0\rangle + b|1\rangle$ and the Hadamard gate acts on each turn by turn.