PH-104 **Modern Physics** 16 April 2024

Solving the Angular part of the Schrodinger Equation

The time independent Schrodinger Equation is,

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) \equiv E\psi(r,\theta,\phi).$$

We assume a solution of the form,

$$\psi(r,\theta,\phi) \equiv R(r)\cdot\Theta(\theta)\cdot\Phi(\phi).$$

Using the definition of the Laplacian operator in spherical coordinates,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial}{\partial\phi}$$

the Schrodinger equation becomes,

$$\begin{aligned}
-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R\Theta\Phi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)R\Theta\Phi + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}R\Theta\Phi\right] + V(r)R\Theta\Phi &\equiv ER\Theta\Phi \\
-\frac{\hbar^2}{2m}\left[\Theta\Phi\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\} + \frac{R\Phi}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right) + \frac{R\Theta}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\Phi\right)\right] + V(r)R\Theta\Phi &\equiv ER\Theta\Phi \\
-\frac{\hbar^2}{2m}\left[\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}\right] + V(r) &\equiv E.
\end{aligned}$$

Multiplying both sides by $\frac{-2mr^2\sin^2\theta}{\hbar^2}$, we obtain,

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \equiv -\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{2mr^2\sin^2\theta}{\hbar^2}\left[V(r) - E\right],$$

where the equation has been adjusted such that the R.H.S is dependent on r and θ, while L.H.S is dependent on ϕ only. Since both sides are equal, they must equal a common separation constant, say $-{m_l}^2$, yielding,

$$\boxed{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -{m_l}^2} \tag{1}$$

$$-\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{2mr^2\sin^2\theta}{\hbar^2}\left[V(r) - E\right] = -{m_l}^2 \tag{2}$$

Now consider equation (2),

$$-\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{2mr^2\sin^2\theta}{\hbar^2}\left[V(r) - E\right] = -{m_l}^2$$

Dividing the equation by $\sin^2\theta$, we obtain,

$$-\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{2mr^2}{\hbar^2}\Big[V(r)-E\Big]=-\frac{{m_l}^2}{\sin^2\theta}$$
$$\frac{{m_l}^2}{\sin^2\theta}-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{2mr^2}{\hbar^2}\Big[V(r)-E\Big].$$

This time, the L.H.S is dependent on θ only, while R.H.S is dependent on r only. Put both sides equal to a common separation constant, say $l(l+1)$. Thus we obtain two additional equations corresponding to the polar and radial parts,

$$\boxed{\frac{{m_l}^2}{\sin^2\theta}-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=l(l+1)} \tag{3}$$

$$\boxed{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{2mr^2}{\hbar^2}\Big[V(r)-E\Big]=l(l+1)}. \tag{4}$$

(1), (3) and (4) are respectively, the azimuthal, polar and radial equations. Notice that the partial derivatives have been replaced by ordinary derivatives $\frac{d}{dr}, \frac{d}{d\theta}, \frac{d}{d\phi}$ etc.

Solving the Azimuthal equation (1) **:**

Equation (1) can be written as,

$$\frac{d^2\Phi}{d\phi^2}+{m_l}^2\Phi = 0.$$

To solve this equation we write the auxiliary equation,

$$\begin{aligned} D^2+{m_l}^2 &= 0 \\ D &= \pm i m_l, \end{aligned}$$

Hence,

$$\Phi(\Phi)=e^{im_l\phi}$$

Actually there are two solutions, $e^{+im_l\phi}$ and $e^{-im_l\phi}$, but we'll cover the latter by allowing m_l to run negative. Here we consider the constraint that the eigenfunctions must be single valued. Since the azimuthal angles $\phi=0$ and $\phi=2\pi$ are the same angles (because when ϕ advances by 2π, we return to the same point in space), it is natural to require that,

$$\begin{aligned} \Phi(\phi) &= \Phi(\phi+2\pi) \\ e^{im_l\phi} &= e^{im_l(\phi+2\pi)} \\ &= e^{im_l\phi}\cdot e^{i2m_l\pi} \\ 1 &= e^{i2m_l\pi} \\ 1 &= \cos(2m_l\pi)+i\sin(2m_l\pi) \end{aligned}$$

From the last expression it follows that m_l must be an integer, i.e.,

$$m_l = 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots.$$

<u>**Solving the Polar equation** (3) **:**</u>

$$-\frac{{m_l}^2}{\sin^2\theta} + \frac{1}{\sin\theta\cdot\Theta}\cdot\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

Multiplying throughout by Θ,

$$\begin{aligned} -\frac{{m_l}^2\Theta}{\sin^2\theta} + \frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) &= -l(l+1)\Theta \\ \frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{{m_l}^2}{\sin^2\theta}\right]\Theta &= 0. \end{aligned}$$

Now, multiplying by $\sin^2\theta$, we obtain,

$$\sin\theta\cdot\cos\theta\frac{d\Theta}{d\theta} + \sin^2\theta\frac{d^2\Theta}{d\theta^2} + \left[\sin^2\theta\cdot l(l+1) - {m_l}^2\right]\Theta = 0. \tag{5}$$

We perform the substitution,

$$\begin{aligned} \cos\theta &= z. \\ \Rightarrow \sin^2\theta &= 1 - z^2 \\ \sin\theta &= \sqrt{1-z^2}. \\ \frac{dz}{d\theta} &= -\sin\theta. \end{aligned}$$

Now if $\Theta(\theta) = P(z)$, P being a function of z, we can use the chain rule to compute and express Θ and θ in terms of P and z,

$$\begin{aligned} \frac{d\Theta}{d\theta} &= \frac{d\Theta}{dz}\cdot\frac{dz}{d\theta} \\ &= -\frac{dP}{dz}\sin\theta \\ &= -\frac{dP}{dz}\sqrt{1-z^2} \\ \Rightarrow \frac{dP}{dz} &= -\frac{1}{\sin\theta}\cdot\frac{d\Theta}{d\theta}. \end{aligned}$$

Furthermore,

$$
\begin{aligned}
\frac{d^2\Theta}{d\theta^2} &= \frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\right) \\
&= \frac{d}{d\theta}\left(-\frac{dP}{dz}\sin\theta\right) \\
&= \frac{d}{dz}\left(-\frac{dP}{dz}\sin\theta\right)\cdot\frac{dz}{d\theta} \\
&= \frac{d}{dz}\left(-\frac{dP}{dz}\sin\theta\right)\cdot -\sin\theta \\
&= \sin\theta\left(\frac{d^2P}{dz^2}\sin\theta + \frac{dP}{dz}\cdot\frac{d}{dz}\sin\theta\right) \\
&= \sin^2\theta\frac{d^2P}{dz^2} + \sin\theta\cdot\frac{dP}{dz}\cdot\frac{d}{dz}\sin\theta
\end{aligned}
$$

Since,

$$
\begin{aligned}
\sin\theta &= \sqrt{1-z^2} \\
\Rightarrow \frac{d}{dz}\sin\theta &= \frac{d}{dz}\left(\sqrt{1-z^2}\right) \\
&= -\frac{z}{\sqrt{1-z^2}},
\end{aligned}
$$

Thus $\frac{d^2\Theta}{d\theta^2}$ becomes,

$$
\begin{aligned}
\frac{d^2\Theta}{d\theta^2} &= (1-z^2)\frac{d^2P}{dz^2} + \sqrt{1-z^2}\cdot\frac{dP}{dz}\cdot -\frac{z}{\sqrt{1-z^2}} \\
&= (1-z^2)\frac{d^2P}{dz^2} - z\frac{dP}{dz}.
\end{aligned}
$$

As a result the polar equation (5) assumes the form,

$$
\begin{aligned}
\sqrt{1-z^2}\cdot z\left(-\frac{dP}{dz}\sqrt{1-z^2}\right) + (1-z^2)\left[(1-z^2)\frac{d^2P}{dz^2} - z\cdot\frac{dP}{dz}\right] + \left[(1-z^2)l(l+1) - {m_l}^2\right]P(z) = 0 \\
-(1-z^2)\cdot z\frac{dP}{dz} + (1-z^2)^2\frac{d^2P}{dz^2} - (1-z^2)z\cdot\frac{dP}{dz} + \left[(1-z^2)l(l+1) - {m_l}^2\right]P(z) = 0 \\
(1-z^2)^2\frac{d^2P}{dz^2} - 2(1-z^2)\cdot z\frac{dP}{dz} + \left[(1-z^2)l(l+1) - {m_l}^2\right]P(z) = 0.
\end{aligned}
$$

Divide by $(1-z^2)$ to obtain,

$$
(1-z^2)\frac{d^2P}{dz^2} - 2z\frac{dP}{dz} + \left[l(l+1) - \frac{m_l^2}{(1-z^2)}\right]P = 0.
$$

This is the historically famous Legendre equation. This equation may also be written in the compact form,

$$
\boxed{\frac{d}{dz}\left[(1-z^2)\cdot\frac{dP}{dz}\right] + \left[l(l+1) - \frac{{m_l}^2}{(1-z^2)}\right]P(z) = 0}. \tag{6}
$$

This ordinary differential equation is frequently encountered in physics and other technical fields. In particular it occurs when solving Laplace's equation and related partial differential equations in spherical coordinates. The Legendre equation may be solved using the standard power series method. Let us perform (yet another) substitution,

$$\begin{aligned} P(z) &= (1-z^2)^{|m_l|/2}G(z), \\ \Rightarrow \frac{dP}{dz} &= \frac{|m_l|}{2}(1-z^2)^{\frac{|m_l|}{2}-1}(-2z)G(z) + (1-z^2)^{\frac{|m_l|}{2}}G^{'}(z) \\ &= -|m_l|z(1-z^2)^{\frac{|m_l|}{2}-1}G(z) + (1-z^2)^{\frac{|m_l|}{2}}G^{'}(z). \end{aligned}$$

Multiplying both sides by $(1-z^2)$, we get,

$$(1-z^2)\frac{dP}{dz} = -|m_l|z(1-z^2)^{\frac{|m_l|}{2}}G(z) + (1-z^2)^{\frac{|m_l|}{2}+1}G^{'}(z).$$

Now differentiate both sides with respect to z to obtain,

$$\begin{aligned} \frac{d}{dz}\left[(1-z^2)\frac{dP}{dz}\right] &= \frac{d}{dz}\left[-|m_l|z(1-z^2)^{\frac{|m_l|}{2}}G(z) + (1-z^2)^{\frac{|m_l|}{2}+1}G^{'}(z)\right] \\ &= -|m_l|(1-z^2)^{\frac{|m_l|}{2}}G(z) - |m_l|z(1-z^2)^{\frac{|m_l|}{2}-1}\frac{|m_l|}{2}(-2z)G(z) - |m_l|z(1-z^2)^{\frac{|m_l|}{2}}G'(z) \\ &+ \left(\frac{|m_l|}{2}+1\right)(1-z^2)^{\frac{|m_l|}{2}}(-2z)G^{'}(z) + (1-z^2)^{\frac{|m_l|}{2}+1}G^{''}(z) \\ &= -|m_l|(1-z^2)^{\frac{|m_l|}{2}}G(z) - |m_l|^2z^2(1-z^2)^{\frac{|m_l|}{2}-1}G(z) - |m_l|z(1-z^2)^{\frac{|m_l|}{2}}G'(z) \\ &- 2z\left(\frac{|m_l|}{2}+1\right)(1-z^2)^{\frac{|m_l|}{2}}G^{'}(z) + (1-z^2)^{\frac{|m_l|}{2}+1}G^{''}(z). \end{aligned}$$

Inserting the value of $P(z)$ and $\frac{d}{dz}\left[(1-z^2)\frac{dP}{dz}\right]$ back into equation (6) we obtain,

$$\begin{aligned} &\left[-|m_l|(1-z^2)^{\frac{|m_l|}{2}}G(z) - |m_l|^2z^2(1-z^2)^{\frac{|m_l|}{2}-1}G(z) - |m_l|z(1-z^2)^{\frac{|m_l|}{2}}G'(z)\right. \\ &\left.-2z\left(\frac{|m_l|}{2}+1\right)(1-z^2)^{\frac{|m_l|}{2}}G^{'}(z) + (1-z^2)^{\frac{|m_l|}{2}+1}G^{''}(z)\right] + \left[l(l+1) - \frac{|m_l|^2}{(1-z^2)}\right](1-z^2)^{|m_l|/2}G(z) = 0 \\ &(1-z^2)^{\frac{|m_l|}{2}+1}G^{''}(z) + \left[-|m_l|z(1-z^2)^{\frac{|m_l|}{2}} - 2z\left(\frac{|m_l|}{2}+1\right)(1-z^2)^{\frac{|m_l|}{2}}\right]G'(z) \\ &+\left[-|m_l|(1-z^2)^{\frac{|m_l|}{2}} + |m_l|^2z^2(1-z^2)^{\frac{|m_l|}{2}-1} + \left(l(l+1) - \frac{|m_l|^2}{(1-z^2)}\right)(1-z^2)^{|m_l|/2}\right]G(z) = 0 \end{aligned}$$

Dividing both sides by $(1-z^2)^{\frac{m_l}{2}}$,

$$\begin{aligned} (1-z^2)G^{''}(z) + \left[-|m_l|z - 2z\left(\frac{|m_l|}{2}+1\right)\right]G'(z) + \left[-|m_l| + \frac{|m_l|^2z^2}{(1-z^2)} + \left\{l(l+1) - \frac{|m_l|^2}{(1-z^2)}\right\}\right]G(z) &= 0 \\ (1-z^2)G^{''}(z) + \left[-2|m_l|z - 2z\right]G'(z) + \left[l(l+1) - |m_l| - \frac{|m_l|^2}{(1-z^2)}(z^2-1)\right]G(z) &= 0 \end{aligned}$$

$$(1-z^2)G^{''}(z) - 2z(|m_l|+1)G'(z) + \left[l(l+1) - |m_l| - |m_l|^2\right]G(z) \quad = \quad 0. \qquad (7)$$

This is the equation we set to solve. We have to find a $G(z)$ that solves (7) and back substitute it to find $P(z)$ and ultimately, $\Theta(\theta)$. Suppose $G(z)$ is a power series (a polynomial in z),

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} a_n z^n \\ G^{'}(z) &= \sum_{n=0}^{\infty} n a_n z^{n-1} \\ G^{''}(z) &= \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2}. \end{aligned}$$

Substitution of these values in Equation 7 yields,

$$\begin{aligned} (1-z^2)\sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} - 2z(|m_l|+1)\sum_{n=0}^{\infty} n a_n z^{n-1} + [l(l+1) - |m_l| - |m_l|^2]\sum_{n=0}^{\infty} a_n z^n = 0 \\ (1-z^2)\sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} - 2(|m_l|+1)\sum_{n=0}^{\infty} n a_n z^{n} + [l(l+1) - |m_l| - |m_l|^2]\sum_{n=0}^{\infty} a_n z^n = 0 \\ (1-z^2)(2a_2 + 3\cdot 2\cdot a_3 z + 4\cdot 3\cdot a_4 z^2 + 5\cdot 4\cdot a_5 z^3 + \ldots) - 2(|m_l|+1)(a_1 z + 2a_2 z^2 + \ldots) \\ +[l(l+1) - |m_l| - |m_l|^2](a_o + a_1 z + a_2 z^2 + \ldots) = 0 \end{aligned}$$

$$\begin{aligned} (2a_2 + 3\cdot 2\cdot a_3 z + 4\cdot 3\cdot a_4 z^2 + 5\cdot 4\cdot a_5 z^3 + \ldots) - (2a_2 z^2 + 3\cdot 2\cdot a_3 z^3 + 4\cdot 3\cdot a_4 z^4 + 5\cdot 4\cdot a_5 z^5 + \ldots) \\ -2(|m_l|+1)(a_1 z + 2a_2 z^2 + \ldots) + [l(l+1) - |m_l| - |m_l|^2](a_o + a_1 z + a_2 z^2 + \ldots) = 0 \quad (8) \end{aligned}$$

leading to,

$$\begin{aligned} (2a_2 + 3\cdot 2\cdot a_3 z + 4\cdot 3\cdot a_4 z^2 + 5\cdot 4\cdot a_5 z^3 + \ldots) &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n \\ (2a_2 z^2 + 3\cdot 2\cdot a_3 z^3 + 4\cdot 3\cdot a_4 z^4 + 5\cdot 4\cdot a_5 z^5 + \ldots) &= \sum_{n=0}^{\infty} n(n-1)a_n z^n \\ (a_1 z + 2a_2 z^2 + \ldots) &= \sum_{n=0}^{\infty} n a_n z^n \\ (a_o + a_1 z + a_2 z^2 + \ldots) &= \sum_{n=0}^{\infty} a_n z^n. \end{aligned}$$

Therefore, Equation (8) can be written as,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n - \sum_{n=0}^{\infty} n(n-1)a_n z^n - 2(|m_l|+1)\sum_{n=0}^{\infty} n a_n z^n + \left[l(l+1) - |m_l| - |m_l|^2\right]\sum_{n=0}^{\infty} a_n z^n = 0. \quad (9)$$

The coefficients of z^n are,

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2(|m_l|+1)n a_n + \left[l(l+1) - |m_l| - |m_l|^2\right]a_n = 0$$

For each term in (8) to be zero, each and every one of these coefficients must be identically zero, resulting in the recursive formula,

$$\begin{aligned}
a_{n+2} &= \left[\frac{n(n-1)+2n(|m_l|+1)-[l(l+1)-|m_l|-|m_l|^2]}{(n+2)(n+1)}\right]a_n \\
&= \left[\frac{n^2-n+2n|m_l|+2n-l(l+1)+|m_l|+|m_l|^2}{(n+2)(n+1)}\right]a_n \\
&= \left[\frac{n^2+n+2n|m_l|-l(l+1)+|m_l|+|m_l|^2}{(n+2)(n+1)}\right]a_n \\
&= \left[\frac{n^2+n+2n|m_l|-l(l+1)+|m_l|+|m_l|^2}{(n+2)(n+1)}\right]a_n \\
&= \left[\frac{(n+|m_l|)(n+|m_l|+1)-l(l+1)}{(n+2)(n+1)}\right]a_n
\end{aligned}$$

Where n, m_l are integers. Since $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is an infinite series, and $z = \cos\theta$, the entire range of z is $(-1 \leq z \leq 1)$ including $z \pm 1$. When z assumes the value ± 1, the series diverges and $G(z)$ becomes infinite. This is not physically possible, because then the wave functions $\Theta(\theta)$ blows up. In other words, we cannot let a_n's go on up to a_∞, otherwise the series G will diverge. A value for $n = 0, 1, 2, 3, \ldots$ reaches, (let's call it n_{max}) for which,

$$(n_{max}+|m_l|)(n_{max}+|m_l|+1) = l(l+1)$$

Since n and m_l are integers, so l is also an integer. There always exists an n_{max}, such that, $l = n+|m_l|$. The possible values of l are $(|m_l|), (|m_l|+1), (|m_l|+2), ..., (|m_l|+n_{max})$, ensuring that the series converges. The constraint $l = n_{max}+|m_l|$ also indicates that $|m_l| \leq l$.

Let us summarize our main results so far.

$$l = n_{max}+|m_l| \tag{10}$$

Don't confuse n_{max} with the principal quantum number n.

$$G(z) = \sum_{n=0}^{n_{max}} a_n z^n \tag{11}$$

$$P(z) = (1-z^2)^{|m_l|/2} G(z) \tag{12}$$

$$\Theta(\theta) = P(z) \tag{13}$$

$$a_{n+2} = \left[\frac{(n+|m_l|)(n+|m_l|+1)-l(l+1)}{(n+2)(n+1)}\right]a_n \tag{14}$$

Example:

Case I: When $l = 0$, $m_l = 0$

If $l = 0$ and $m_l = 0$, then from equation (10), $n_{max} = 0$.
Thus $G(z)$ will be,

$$\begin{aligned}
G(z) &= a_o \\
P(z) &= (1-z^2)^0 a_o = a_0 \\
\Theta(\theta) &= P(z) \propto 1
\end{aligned}$$

Case II: When $l = 1$, $|m_l| = 0, 1$, $\Rightarrow m_l = 0, \pm 1$

(a) Consider $l = 1$, and $m_l = 0$, then from equation (9), $n_{max} = 1$.

$$G(z) = a_o + a_1 z$$

From equation (10) it is clear that, $a_2 = 0$, because $n_{max} = 1$ and $a_o = 0$, thus

$$\begin{aligned} G(z) &= a_1 z \\ P(z) &= (1 - z^2)^0 a_1 z = a_1 z \\ \Theta(\theta) &\propto z = \cos\theta \end{aligned}$$

(b) Consider $l = 1$, and $m_l = 1$, then from equation (10), $n_{max} = 0$.

$$\begin{aligned} G(z) &= a_o \\ P(z) &= (1 - z^2)^{1/2} a_o = a_o (1 - z^2)^{1/2} \\ \Theta(\theta) &\propto (1 - z^2)^{1/2} = \sin\theta \end{aligned}$$

(c) Consider $l = 1$, and $m_l = -1$, then from equation (10), $n_{max} = 2$.

$$G(z) = a_o + a_1 z + a_2 z^2$$

From equation (10), $a_2 = -a_o$,

$$\begin{aligned} G(z) &= a_o + a_1 z - a_o z^2 \\ &= a_o (1 - z^2) + a_1 z \end{aligned}$$

Since $n_{max} = 2$, so $a_3 = 0, \Rightarrow a_1 = 0$.

$$\begin{aligned} G(z) &= a_o (1 - z^2) \\ P(z) &= (1 - z^2)^{-1/2} a_o (1 - z^2) \\ &= a_0 (1 - z^2)^{1/2} \\ \Theta(\theta) &\propto (1 - z^2)^{1/2} = \sin\theta \end{aligned}$$

Case III: When $l = 2$, $|m_l| = 0, 1, 2$, $\Rightarrow m_l = 0, \pm 1, \pm 2$

(a) Consider $l = 2$, and $m_l = 0$, then from equation (10), $n_{max} = 2$.

$$G(z) = a_o + a_1 z + a_2 z^2$$

Where $a_2 = -3a_o$. Also $a_3 = 0, \Rightarrow a_1 = 0$.

$$\begin{aligned} G(z) &= a_o - 3a_o z^2 \\ &= a_o (1 - 3z^2) \\ P(z) &= (1 - z^2)^{0/2} a_o (1 - 3z^2) \\ &= a_o (1 - 3z^2) \\ \Theta(\theta) &\propto (1 - 3z^2) = (1 - 3\cos^2\theta) \end{aligned}$$

(b) Consider $l = 2$, and $m_l = 1$, then from equation (10), $n_{max} = 1$.

$$G(z) = a_o + a_1 z$$

Where $a_2 = -2a_o$. Since due to $n_{max} = 1$, we have $a_2 = 0$, $\Rightarrow a_o = 0$.

$$\begin{aligned} G(z) &= a_1 z \\ P(z) &= a_1 z(1-z^2)^{1/2} \\ \Theta(\theta) &\propto z(1-z^2)^{1/2} = \sin\theta\cos\theta. \end{aligned}$$

(c) Consider $l = 2$, and $m_l = 2$, then from equation (10), $n_{max} = 0$.

$$\begin{aligned} G(z) &= a_o \\ P(z) &= a_o(1-z^2) \\ \Theta(\theta) &\propto (1-z^2) = \sin^2\theta. \end{aligned}$$

Case IV: When $l = 3$, $|m_l| = 0, 1, 2, 3$, $\Rightarrow m_l = 0, \pm 1, \pm 2, \pm 3$

(a) Consider $l = 3$, and $m_l = 0$, then from equation (10), $n_{max} = 3$.

$$G(z) = a_o + a_1 z + a_2 z^2 + a_3 z^3$$

Where $a_2 = -6a_o$ and $a_3 = -\frac{5}{3}a_1$. Since $a_o = 0$ otherwise series diverges, hence $a_2 = 0$. Therefore,

$$\begin{aligned} G(z) &= a_1 z - \frac{5}{3}a_1 z^3 \\ &= a_1(3z - 5z^3) \\ P(z) &= a_1(3z - 5z^3) \\ \Theta(\theta) &\propto (3z - 5z^3) = z(3 - 5z^2) = \cos\theta(3 - 5\cos^2\theta) \end{aligned}$$

(b) Consider $l = 3$, and $m_l = 1$, then from equation (10), $n_{max} = 2$.

$$G(z) = a_o + a_1 z + a_2 z^2$$

Where $a_2 = -5a_o$ and $a_3 = -a_1$. Since $n_{max} = 2$, this gives $a_3 = 0 \Rightarrow a_1 = 0$. Hence,

$$\begin{aligned} G(z) &= a_0 - 5a_o z^2 \\ &= a_o(1 - 5z^2) \\ P(z) &= a_o(1-5z^2)(1-z^2)^{1/2} \\ \Theta(\theta) &\propto (1-5z^2)(1-z^2)^{1/2} = (1 - 5\cos^2\theta)\sin\theta \end{aligned}$$

(c) Consider $l = 3$, and $m_l = 2$, $\Rightarrow n_{max} = 1$.

$$G(z) = a_o + a_1 z$$

Where $a_2 = -2a_o$. Also due to $n_{max} = 1$, $a_2 = 0$, which implies $a_o = 0$.

$$\begin{aligned} G(z) &= a_1 z \\ P(z) &= a_1 z(1-z^2) \\ \Theta(\theta) &\propto z(1-z^2) = \cos\theta\sin^2\theta \end{aligned}$$

(d) Consider $l = 3$, and $m_l = 3$, $\Rightarrow n_{max} = 0$.

$$\begin{aligned} G(z) &= a_o \\ P(z) &= a_o(1-z^2)^{3/2} \\ \Theta(\theta) &\propto (1-z^2)^{3/2} = \sin^3\theta \end{aligned}$$