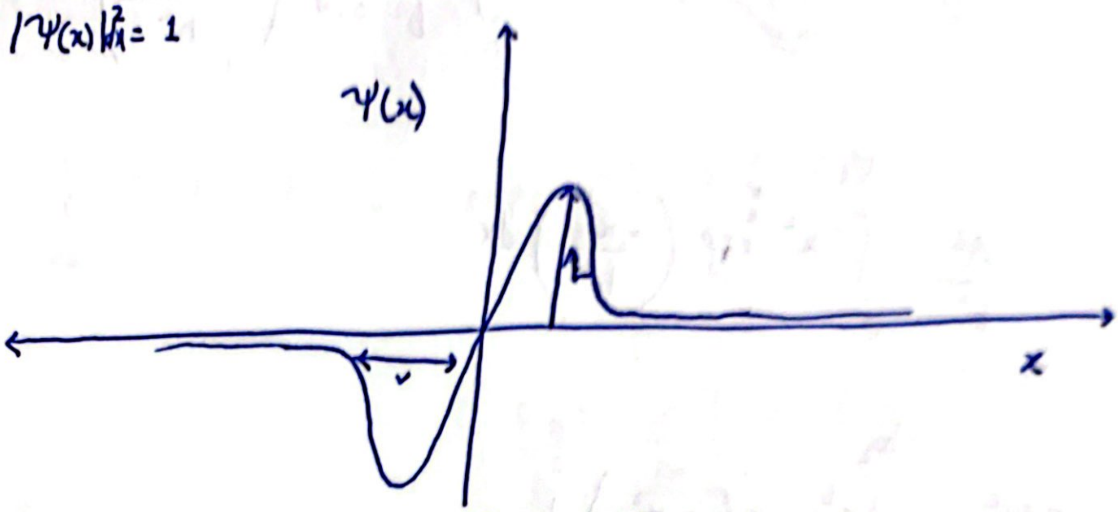
Question 1

(a) $\int_{-\infty}^{\infty} |\Psi(x)|^2 dx = 1$



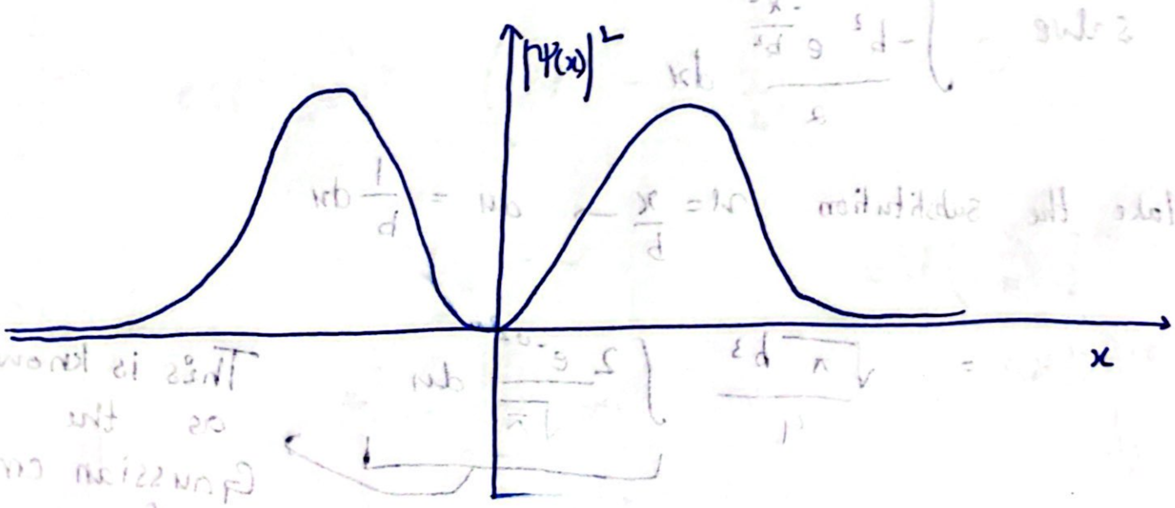
Δx and v depend on A and b.

Try ~~pat~~ plotting for different values of A and b on Desmos!

(b) Probability density $|\Psi(x)|^2 = \left(\frac{Ax}{b} \exp\left\{\frac{-x^2}{2b^2}\right\}\right)^2$

$$= \frac{A^2x^2}{b^2} \exp\left(\frac{-\cancel{2}x^2}{\cancel{2}b^2}\right)$$

Plot this! $\longrightarrow$ $$= \frac{A^2x^2}{b^2} \exp\left(\frac{x^2}{b^2}\right)$$



(c) Probability $= \cancel{\int_{-\infty}^{\infty} \psi(x)}$ $\int_0^{b/2} \frac{A x^2}{b^2} \exp\left(-\frac{x^2}{b^2}\right) dx$

$$= \frac{A^2}{b^2} \int_0^{b/2} x^2 \exp\left(\frac{-x^2}{b^2}\right) dx$$

Let's compute $\int_0^{b/2} x^2 \exp\left(\frac{-x^2}{b^2}\right) dx$

Using Integration by parts $\int f g' = fg - \int f' g$

$f = x \qquad g' = x e^{\frac{-x^2}{b^2}}$

$f' = 1 \qquad g = -\frac{b^2 e^{\frac{-x^2}{b^2}}}{2}$

$$= -\frac{b^2 x e^{-\frac{x^2}{b^2}}}{2} - \int \frac{-b^2 e^{\frac{-x^2}{b^2}} dx}{2}$$

Let's salve $\int \frac{-b^2 e^{\frac{-x^2}{b^2}}}{2} dx$

Make the substitution $u = \frac{x}{b} \rightarrow du = \frac{1}{b} dx$

$$= \frac{\sqrt{\pi}\, b^3}{4} \underbrace{\int \frac{2 e^{-u^2}}{\sqrt{\pi}} du}_{\text{erf} u}$$

This is known as the Gaussian error function

We get $\rightarrow \frac{-\sqrt{\pi} b^3 \operatorname{erf}(u)}{4}$

Plug $u = \frac{x}{b}$

$$= \frac{-\sqrt{\pi}\, b^3 \operatorname{erf}\left(\frac{x}{b}\right)}{4}$$

we get $\frac{-b^2 x e^{\frac{-x^2}{b^2}}}{2} - \int -\frac{b^2 e^{\frac{-x^2}{b^2}}}{2} du$

$$= \frac{\sqrt{\pi}\, b^3 \operatorname{erf}\left(\frac{x}{b}\right)}{4} - \frac{b^2 x e^{-\frac{x^2}{b^2}}}{2}$$

Simplify and we get

$$\left. \frac{A^2 \left(\sqrt{\pi}\, b \operatorname{erf}\left(\frac{x}{b}\right) - 2x e^{\frac{-x^2}{b^2}}\right)}{4} \right|_0^{b/2}$$

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Question 1

(d) $|\psi(x)|^2 = \dfrac{A^2x^2}{b^2} \exp\left(-\dfrac{x^2}{b^2}\right)$

$$\frac{d}{dx}|\psi(x)|^2 = 0 \rightarrow \exp\left(\frac{-x^2}{b^2}\right)\frac{d}{dx}\left[\frac{A^2x^2}{b^2}\right] + \frac{A^2x^2}{b^2}\frac{d}{dx}\left[\exp\left(\frac{-x^2}{b^2}\right)\right]$$

$$= \exp\left(-\frac{x^2}{b^2}\right)\frac{2A^2x}{b^2} + \frac{A^2x^2}{b^2}\cdot\frac{-2x}{b^2}\exp\left(\frac{-x^2}{b^2}\right) = 0$$

↓ divide both sides with $\cancel{\frac{2A^2}{b^2}}$ $\frac{b^2}{2A^2}$

$$\cancel{= \exp\left(\frac{-x^2}{b^2}\right)\left[\frac{2A^2x}{b^2}\left(1 \bullet \frac{2x}{b^2}\right)\right] = 0}$$

$$\Rightarrow \exp\left(\frac{-x^2}{b^2}\right)x - \frac{x^3}{b^2}\exp\left(\frac{-x^2}{b^2}\right) = 0$$

$$\exp\left(\frac{-x^2}{b^2}\right)x\left(1 - \frac{x^2}{b^2}\right) = 0$$

$$1 - \frac{x^2}{b^2} = 0 \qquad \exp\left(\frac{-x^2}{b^2}\right)x = 0$$

$$x^2 = b^2 \qquad \cancel{x = 0}$$

$$\boxed{x = \pm b}$$

↓

Question 2

$$E = \frac{hc}{\lambda}$$

$h = 6.626 \times 10^{-34}\ J.s$
$c = 3 \times 10^{8}\ m/s$
$eV = 1.602 \times 10^{-19}\ J$

$$\lambda = \frac{hc}{20eV} =$$

Question 3

→ Let's write down the Schrodinger equation ~~s~~ in 3-D.

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\Psi(x,y,z) = [V(x,y,z) - E]\,\Psi(x,y,z)$$

→ $V(x,y,z) = 0$ Inside the well.

→ We also ~~so~~ use separation of variables $\Psi(x,y,z) = XYZ$ and simplify the equation ⇒

$$-\frac{\hbar^2}{2m}\left[YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2}\right] = (XYZ)\, E_x E_y E_z$$

→ dividing both sides by XYZ

$$-\frac{\hbar^2}{2m}\left[\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2}\right] = E_x E_y E$$

We can write $\frac{1}{X}\frac{d^2X}{dX^2} = E_x\left(-\frac{2m}{\hbar^2}\right)$

and so on.

We solve for X, Y and Z in a similar manner

$$\frac{d^2X}{dx^2} = X E_x \cdot \frac{-2m}{\hbar^2}$$

$$k_x^2 = \frac{2 E_x m}{\hbar^2}$$

$$k_x = \frac{\sqrt{2mE_x}}{\hbar}$$

$$\frac{d^2X}{dx^2} = -k^2 X$$

- Solving for this we get

$$X(x) = A \sin kx + B \cos(kx)$$

- Applying Boundary condition $\Psi(x)$ is a continous function.

$$\Psi(0) = \Psi(L) = 0$$ (because of infinite potenti

$$X(L) = A \sin(kL) + B \cos(kL) = 0$$

$$X(0) = A \sin(0) + B \cos(0)$$
$$= B \cos(0) = B = 0$$

$$\cancel{A \sin(kx} \Rightarrow A \sin(kL) = 0 \quad -①$$

To satisfy ① $kL = 0, \pm\pi, \pm 2\pi \pm 3\pi$

$kL = n\pi$ where n is an interg

$$k_x = \frac{n\pi}{L}$$

$\cancel{n = \pm 1}$, 2

$$\cancel{k_x = \frac{\sqrt{2mE_x}}{\hbar}} \qquad \cancel{E_x = \frac{k_x^2 \hbar^2}{2m}}$$

$n = 1, 2, 3, 4, 5.$

$$\cancel{\frac{n^2\pi^2}{L^2}} = \frac{2m E_x}{\hbar^2} \qquad E_x = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$$X(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_x \pi x}{L}\right)$$

~~Ex~~ To find E_x

$$k_x^2 = \frac{n^2 \pi^2}{L^2} = \frac{2m E_x}{\hbar^2}$$

$$E_x = \frac{n_x^2 \pi^2 \hbar^2}{2mL^2}$$

Similarly $$Y(y) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_y \pi y}{L}\right)$$

$$E_y = \frac{n_y^2 \pi^2 \hbar^2}{2mL^2}$$

$$Z(z) = \sqrt{\frac{2}{F}} \sin\left(\frac{n_z \pi z}{F}\right)$$

$$\cancel{E_z = \sqrt{\frac{2}{F}}} \quad E_z = \frac{n_z^2 \pi^2 \hbar^2}{2mF^2}$$

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(C) Ground state energy levels are non-degenerate

→ Only one possibility: $n_x = 1$ $n_y = 1$ $\wedge$ $n_z = 1$

$$E = E_x + E_y + E_z = \frac{\pi^2\hbar^2}{2mL^2} + \frac{\pi^2\hbar^2}{2mL^2} + \frac{\pi^2\hbar^2}{2mF^2}$$

☆ For the 1st Excited state, let's first consider the case ; $\boxed{L > F}$

$n_x = 1$ $n_y = 2$ $n_z = 1$ ~~will~~ gives us the next excited state.

$$\frac{\pi^2\hbar^2}{2mL^2} + \frac{2\pi^2\hbar^2}{2mL^2} + \frac{\pi^2\hbar^2}{2mF}$$

→ It has one degeneracy because another possible way of getting the same energy is:

$n_x = 2$ $n_y = 1$ $n_z = 1$

☆ Now, let's consider the case where $L < F$.

In this case the ~~next high~~ 1st excited state energy is given by

$n_x = 1$, $n_y = 1$ and $n_z = 2$

Again, for the second excited state let's consider the simple case where $L > F$.

~~To~~ $n_x = 1$ $n_y = 1$ $n_z = 2$ ~~will give~~

When $L < F$, the next energy level is given by $n_x = 2$ $n_y = 1$ $n_z = 1$

$n_x = 1$ $n_y = 2$ $n_z = 1$

→ degeneracy = 2 atleast!

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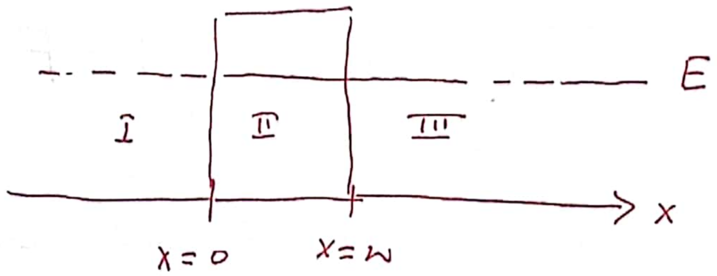


In regions I, III $V=0$ so $\psi(x)$ is such that

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \Rightarrow \quad \boxed{\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi}$$

So $\psi(x) = Ae^{ik_1x} + Be^{-ik_1x}$ in I.

& $\psi(x) = Ce^{ik_1x} + De^{-ik_1x}$ in III with

$$k_1 = +\sqrt{\frac{2mE}{\hbar^2}}.$$

We take a particle coming in from the left, so $D=0$.

In region II, $\psi(x)$ is such that

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}.$$

Note that $E<V_0$ so $(E-V_0)<0$

so $-(E-V_0)>0$.

So our solution is

$$\psi(x) = Ee^{k_2x} + Fe^{-k_2x} \quad \text{with } k_2 = +\sqrt{\frac{2m(V_0-E)}{\hbar^2}}.$$

What do we know about $A, B, C, E, \& F$?

at $x = 0$

$$A + B = E + F \quad (\text{continuity of } \psi(x)). \quad 1$$

and

$$\cancel{ik_1x} \; ik_1 A - ik_1 B = k_2 E - k_2 F \quad \left(\text{continuity of } \frac{d\psi}{dx}\right). \quad 2$$

at $x = W$

$$Ce^{ik_1W} + \cancel{De^{-ik_1W}} = Ee^{k_2W} + Fe^{-k_2W}. \quad 3$$

and

$$ik_1 Ce^{ik_1W} \cancel{-ik_1 De^{-ik_1W}} = k_2 Ee^{k_2W} - k_2 Fe^{-k_2W}. \quad 4$$

The probability of transmission is given by $\left|\frac{C}{A}\right|^2$.

So we try to find $\frac{C}{A}$.

We will make $C \& A$ talk through E, F.

Eq. 2 can be written as

$$A - B = -i\frac{k_2}{k_1}E + i\frac{k_2}{k_1}F$$

You can find a nice (latex-ed) version of this working on Brilliant.

1 & 2 then imply

$$\boxed{2A = E\left(1 - \frac{ik_2}{k_1}\right) + F\left(1 + \frac{ik_2}{k_1}\right).}$$

'A' in terms of $E \& F$.

Now, 4 can be written as

$$i\frac{k_1}{k_2}Ce^{ik_1W} = Ee^{k_2W} - Fe^{-k_2W}.$$

Adding to 3 yields

$$\boxed{2Ee^{k_2W} = Ce^{ik_1W}\left(1 + \frac{ik_1}{k_2}\right)}.$$

'E' in terms of C.

Subtracting 4 & 3 yields

$$\boxed{2Fe^{-k_2w} = Ce^{ik_1w}\left(1-\frac{ik_1}{k_2}\right).} \rightarrow F \text{ in terms of } C.$$

We can now use our expression for A to ~~talk about~~ bring C in (via these expressions for F, E).

$$2A = \left(1-\frac{ik_2}{k_1}\right)\left(1+\frac{ik_1}{k_2}\right)\frac{e^{ik_1w}}{2e^{k_2w}}C +$$

$$\left(1+\frac{ik_2}{k_1}\right)\left(1-\frac{ik_1}{k_2}\right)\frac{e^{ik_1w}}{2e^{-k_2w}}C$$

$$\Rightarrow \frac{4A}{C} = \left(1-\frac{ik_2}{k_1}\right)\left(1+\frac{ik_1}{k_2}\right)e^{ik_1w}e^{-k_2w} +$$

$$\left(1+\frac{ik_2}{k_1}\right)\left(1-\frac{ik_1}{k_2}\right)e^{ik_1w}e^{k_2w}.$$

Since we want $|\cdot|^2$, & a common factor of e^{ik_1w} does nothing (you can see it by dividing it on both sides & then proceeding)

$$\cancel{\Rightarrow 16\left|\frac{A}{C}\right|}$$

$$\frac{4A}{C} = \left(2+\frac{ik_1}{k_2}-\frac{ik_2}{k_1}\right)e^{-k_2w} + \left(2-\frac{ik_1}{k_2}+\frac{ik_2}{k_1}\right)e^{k_2w}.$$

To proceed, look up the defs of cosh(x) and sinh(x).

$$\frac{4A}{C} = 4\cosh(k_2 w) + 2i\left(\frac{k_2^2 - k_1^2}{k_1 k_2}\right)\sinh(k_2 w).$$

$$\Rightarrow \frac{A}{C} = \cosh(k_2 w) + i\left(\frac{k_2^2 - k_1^2}{2k_1 k_2}\right)\sinh(k_2 w).$$

$$\left|\frac{A}{C}\right|^2 = \cosh^2(k_2 w) + \left(\frac{k_2^2 - k_1^2}{2k_1 k_2}\right)^2 \sinh^2(k_2 w).$$

Since $\cosh^2(x) - \sinh^2(x) = 1$

$$\Rightarrow \left|\frac{A}{C}\right|^2 = 1 + \sinh^2\overset{k_2 w}{\cancel{(x)}}\Bigg[\underbrace{1 + \frac{(k_2^2 - k_1^2)^2}{4k_1^2 k_2^2}}\Bigg].$$

$$\downarrow$$

$$\frac{4k_1^2 k_2^2 + k_1^4 + k_2^4 - 2k_1^2 k_2^2}{4k_1^2 k_2^2}$$

$$\downarrow$$

$$\frac{k_1^4 + k_2^4 + 2k_1^2 k_2^2}{4k_1^2 k_2^2}$$

$$\downarrow$$

$$\left|\frac{A}{C}\right|^2 = 1 + \sinh^2\underset{k_2 w)}{\cancel{(x)}}\left(\frac{k_1^2 + k_2^2}{2k_1 k_2}\right)^2$$

$$\Rightarrow \left|\frac{C}{A}\right|^2 = \frac{1}{1 + \sinh^2(k_2 w)\left(\frac{k_1^2 + k_2^2}{2k_1 k_2}\right)^2}.$$

→ We have, in this process, found

→ C in terms of A

→ E, F in terms of C (so also in terms of A).

→ We can find B in terms of A by noting

that $\cancel{A}\ B = E + F - A$.

~~One~~ It is good that we can't find 'A'.

It, in some sense, reps how many particles

are 'coming in' from the left — a physical

condition we haven't specified in this problem.

Solution to Assignment 4

Q.8.a) The diff. eq in the region $x<0$ is

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \Rightarrow \psi(x) = Ae^{ik_1x} + Be^{-ik_1x}$$

Where $k_1 = +\sqrt{\frac{2mE}{\hbar^2}}$.

In the region $x \geq 0$, the ~~relative~~ relevant diff. eq is

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V_0\psi = E\psi$$

$$\Rightarrow \frac{d^2}{dx^2}\psi = -\frac{2m(E-V_0)}{\hbar^2}\psi \Rightarrow \psi(x) = Ce^{ik_2x} + De^{-ik_2x}$$

Where $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$. Note that $(E-V_0)>0$ so, $k_2 \in \mathbb{R}$ still.

So far,

$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x<0 \\ Ce^{ik_2x} + De^{-ik_2x} & x \geq 0. \end{cases}$$

what are A, B, C, D?

If our particle is coming in from the left (and heading to the right), we can set $D=0$. (Read comment)

Now, we require that $\psi(x)$ be continious at all x. $x=0$ is where the two different functions above meet. Continuity @ $x=0$ mandates $A+B=C$.

We also require the derivative of $\psi(x)$ to be continuous at all x. Imposing this at $x=0$ yields (Read comment).

$$ik_1 A - ik_1 B = ik_2 C \Rightarrow ik_1(A-B) = ik_2 C$$

$$\Rightarrow A - B = \left(\frac{k_2}{k_1}\right) C.$$

So far, there are two linear equations for three unknowns. What do we do next?

$$A + B = C \quad -(1)$$

$$A - B = \frac{k_2}{k_1} C \quad -(2)$$

$$\Rightarrow 2A = \left(\frac{k_1}{k_1} + \frac{k_2}{k_1}\right) C$$

$$\Rightarrow C = 2\left(\frac{k_1}{k_1+k_2}\right) A$$

Similarly (1) & (2) imply

$$2B = \left(1 - \frac{k_2}{k_1}\right) C$$

$$\Rightarrow B = \frac{C}{2}\left(\frac{k_1-k_2}{k_1}\right)$$

$$\Rightarrow B = \frac{\cancel{2}}{\cancel{2}}\left(\frac{\cancel{k_1}}{k_1+k_2}\right)\left(\frac{k_1-k_2}{\cancel{k_1}}\right) A$$

$$\Rightarrow B = \left(\frac{k_1-k_2}{k_1+k_2}\right) A.$$

We have both B, C in terms of A. Without further information, we can not determine A. 'A' represents the number of particles incident over some time (in some sense).

b) The diff. equation on the left is the same as part a,

so $\frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\psi \Rightarrow \psi(x) = Ae^{ik_1x} + Be^{-ik_1x}$

with $k_1 = +\sqrt{\frac{2mE}{\hbar^2}}$.

For $x \geq 0$,

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi.$$ Note that

$(E-V_0) < 0$ so $-(E-V_0) > 0 \Rightarrow (V_0 - E) > 0$.

$\psi(x) = \cancel{Ce^{ik_2}}\ Ce^{k_2x} + De^{-k_2x}$. Note that we don't have 'i' in the exponent anymore.

Here $k_2 = +\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$. You should check that the suggested solution works.

$$\text{so } \psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x < 0 \\ Ce^{k_2x} + De^{-k_2x} & x \geq 0. \end{cases}$$

Here, we set $C = 0$. If $C \neq 0$, $\psi(x) \to \infty$ as $x \to \infty$.

Since $|\psi(x)|^2$ gives the probability distribution function for the particle's position, we ~~cannot~~ cannot allow that. This doesn't happen with $D \neq 0$ since $e^{-k_2x} \to 0$ as $x \to \infty$.

$$\text{So } \psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x < 0 \\ De^{-k_2x} & x \geq 0. \end{cases}$$

We now want to find A, B, C

Continuity at $x=0$ mandates $A + B = D$.

Continuity of $\frac{d\psi}{dx}$ at $x=0$ mandate

$$\cancel{ik_1 x} \quad ik_1 A - ik_1 B = -k_2 D.$$

$$\Rightarrow A - B = \frac{-k_2}{ik_1} D$$

$$\Rightarrow A - B = \frac{ik_2}{k_1} D.$$

We proceed similarly as in 8a.

The aforementioned imply

$$2A = D\left(\frac{k_1}{k_1} + \frac{ik_2}{k_1}\right).$$

$$\Rightarrow D = \left(\frac{k_1}{k_1 + ik_2}\right)(2A)$$

Similarly,

$$2B = D\left(\frac{k_1}{k_1} - \frac{ik_2}{k_1}\right)$$

$$\Rightarrow B = \frac{D}{2}\left(\frac{k_1 - ik_2}{k_1}\right)$$

$$\Rightarrow B = A\left(\frac{k_1 - ik_2}{k_1 + ik_2}\right).$$

Comment for Q.8.

In $\psi(x) = Ce^{ik_2x} + De^{-ik_2x}$, we take Ce^{ik_2x} to represent a right moving wave & De^{-ik_2x} to rep. a left moving one. We can't see that without the time dependence. With it,

$$\psi(x,t) = \left(Ce^{ik_2x} + De^{-ik_2x} \right) e^{-i\omega t}, \quad \omega, t \in \mathbb{R}.$$

$$= Ce^{i(k_2x-\omega t)} + De^{-i(k_2x+\omega t)}.$$

Let's look at the real part of each term.

The first one is $C\cos(k_2x-\omega t)$. The second is $D\cos(k_2x+\omega t)$.

C, D are constants. Let's stare at just $\cos(k_2x\pm\omega t)$.

Suppose we are at $x = x_0$ when $t=0$.

$$\cos(k_2x_0+\omega(0)) = \cos(k_2x_0).$$

$$\cos(k_2x_0-\omega(0)) = \cos(k_2x_0).$$

At $t=0$, both waves have the same ~~height~~ displacement (height) at all points.

What if we look at it some time, t_0, later?

Where will the height that is initially found at $x = x_0$ be found now? Well, if the argument of (the thing in the cosine).

$\cos(k_2 x \pm \omega t_0)$ becomes $k_2 x_0$, we'll recover our height.

So, we want

$$k_2 x \pm \omega t_0 = k_2 x_0$$

$$\Rightarrow k_2 x = k_2 x_0 - \omega t_0$$

$$\Rightarrow x = x_0 - \frac{\omega}{k_2} t_0$$

Since $\omega, k_2 > 0$ this $x < x_0$. So for $\cos(k_2 x + \omega t)$ what was initially found at $x = x_0$ is, ~~now~~ later found at a point to its left. This represents a left moving wave.

A similar exercise with $k_2 x - \omega t_0$ should show you that it moves to the right.

Comment for Q.8 contd:

We demand that $\psi(x)$ be:

i) Continuous at all points.

ii) $\dfrac{d\psi}{dx}$ is continuous at all points.

iii) $\displaystyle\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1$.

Please see the tutorial on this for a discussion of why we demand these.

(9) for $x<0$, we can immediately write
(a) down

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x}.$$

with $k_1 = +\sqrt{\frac{2mE}{\hbar^2}}$ since $V=0$ and $E>0$.

for $x \geq 0$, $V(x) = -V_0$ so

our diff. equation is

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E+V_0)}{\hbar^2}\psi$$

Note that $\frac{2m(E+V_0)}{\hbar^2} > 0$ so our solution

becomes

$$\psi(x) = \cancel{A}\, Ce^{ik_2x} + De^{-ik_2x}$$

with $k_2 = +\sqrt{\frac{2m(E+V_0)}{\hbar^2}}$.

Again, we impose that the "particle comes in from the left" so we set $D=0$.

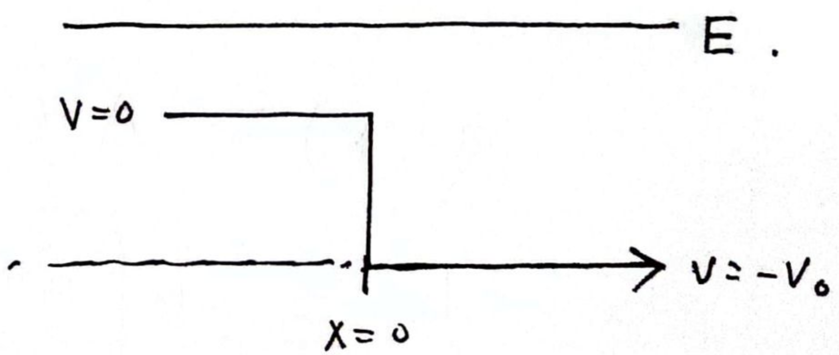
So far,

$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x<0 \\ Ce^{ik_2x} & x \geq 0. \end{cases}$$

We now can find B, C in terms of A just as in 8a.

~~(b) Since E > 0, this energy~~ meets

(b) Note that all that part (a) assumes for the energy is that $E > 0$. Since this is also guarenteed here, the solution is the same. As in, the situation described in both parts is

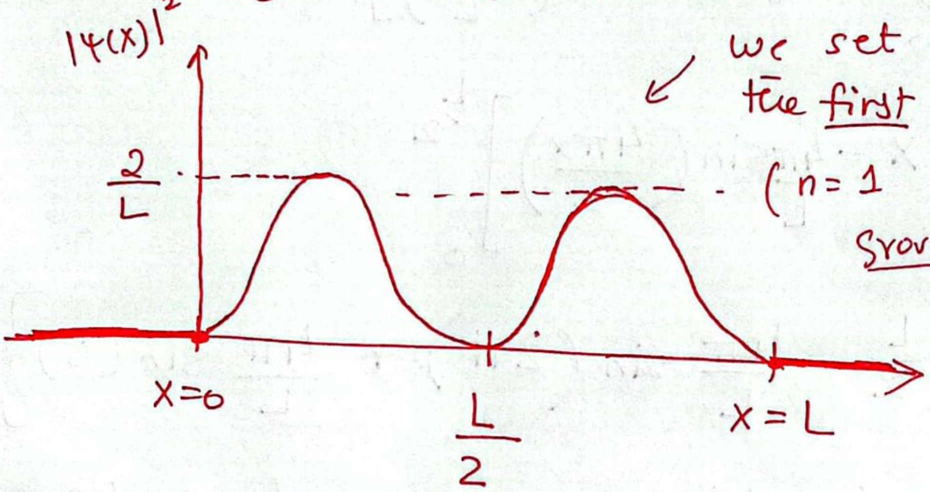


~~9b)~~ 10) a) As we found in class, for the inf. potential well

$$\psi(x) = \begin{cases} 0 & x \leq 0 \\ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 < x < L \\ 0 & x \geq L. \end{cases}$$

The prob density is $|\psi(x)|^2$ so

$$|\psi(x)|^2 = \begin{cases} 0 & x \leq 0 \\ \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) & 0 < x < L \\ 0 & x \geq L. \end{cases}$$



we set $n=2$ for the first excited state ($n=1$ is the ground state).

b) Since the probability outside the well is zero, it must be inside the well, i.e. between $x=0$ & $x=L$. So the probability of being found between 0 & L is 1.

One could also do this by $\int_0^L |\psi(x)|^2 dx$

$$= \int_0^L \frac{2}{L} \sin^2\left(\frac{2\pi x}{L}\right) dx.$$

(c) $$\int_0^{L/2} |\psi(x)|^2\, dx = \int_0^{L/2} \frac{2}{\cancel{\sqrt{}}L} \sin^2\left(\frac{2\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^{L/2} \sin^2\left(\frac{2\pi x}{L}\right) dx = \frac{2}{L}\int_0^{L/2} dx \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{4\pi x}{L}\right)\right]$$

Since $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \quad \forall x \in \mathbb{R}$.

$$= \frac{1}{L}\int_0^{L/2} dx \left[1 - \cos\left(\frac{4\pi x}{L}\right)\right] \cancel{dx}$$

$$= \frac{1}{L}\left[x - \frac{4\pi}{L}\sin\left(\frac{4\pi x}{L}\right)\right]_0^{L/2}$$

$$= \frac{1}{L}\left[\frac{L}{2} - \frac{4\pi}{L}\sin(2\pi) + \frac{4\pi}{L}\sin(0)\right]$$

$= \frac{1}{2}$ (which we expect if you stare at the graph in part a ☺).

(d) As ~~we'd~~ we found in class, $E_n = \frac{\hbar^2\pi^2}{2mL^2} n^2$

$$E_2 \text{ (first excited state)} = \frac{\hbar^2\pi^2}{2mL^2} 2^2 = \frac{4\hbar^2\pi^2}{2mL^2}$$

$$E_3 \text{ (second excited state)} = \frac{\hbar^2\pi^2}{2mL^2} 3^2 = \frac{9\hbar^2\pi^2}{2mL^2}$$

$$E_3 - E_2 = (9-4)\frac{\hbar^2\pi^2}{2mL^2} = \frac{5\hbar^2\pi^2}{2mL^2}.$$

(e) 600 nm corresponds to ~~orange~~ red in the visible spectrum of EM waves.

The energy of the photon is $\cancel{\frac{\hbar f}{2\pi}}$ $\hbar(2\pi f)$.

Now since $c = (\lambda)(f)$

speed of the wave.

$$\frac{c}{\lambda} = f \quad \text{so} \quad E = \frac{\hbar 2\pi c}{\lambda}.$$

This E corresponds to $E_3 - E_2 = \frac{5\hbar^2\pi^2}{2mL^2}$.

So then,

$$\frac{5\hbar^2\pi^2}{2mL^2} = \frac{2\cancel{\hbar}\cancel{\pi}c}{\lambda} \Rightarrow L = \sqrt{\frac{5\hbar\pi\lambda}{4cm}}$$

if we use SI units for all quantities on the right, we get the length of the well in metre. This length depends on the mass of the particle.

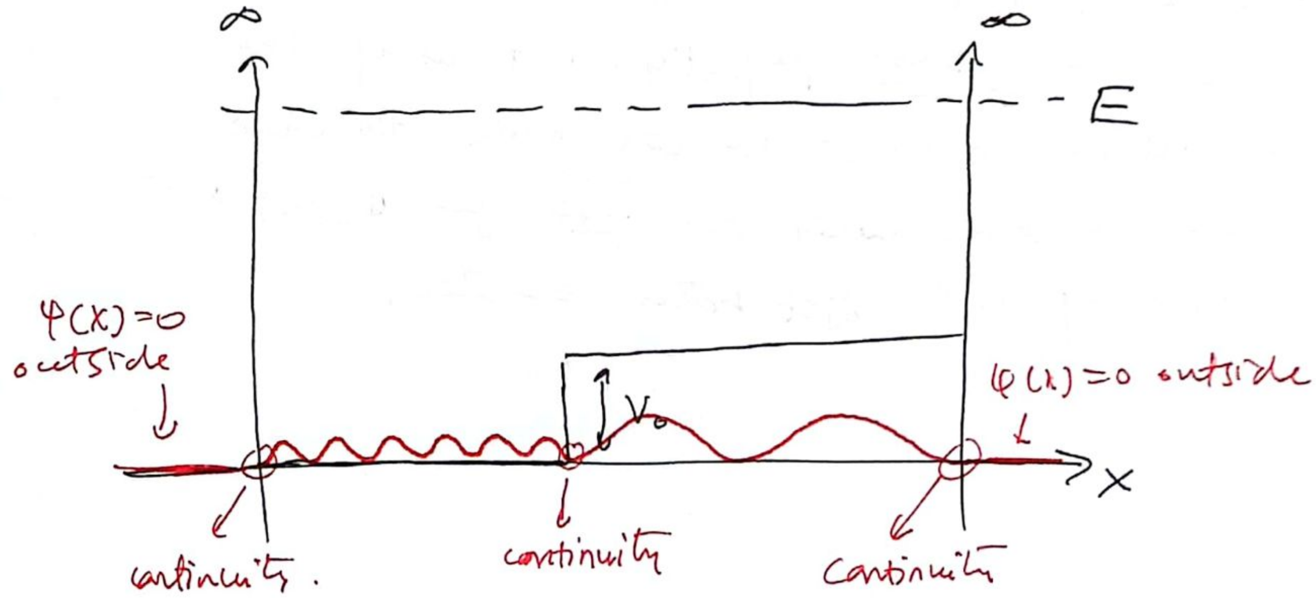
(1) In the well,

$\psi(x) = Ae^{ikx} + Be^{-ikx}$ with $k = +\sqrt{\frac{2m(E-V)}{\hbar^2}}$ as long as $E > V$ (which is guaranteed here).

The larger $(E-V)$ is, the larger 'k' is which implies that we have smaller wavelengths (try plotting $\sin(30x)$ & $\sin(3x)$).

Similarly, the larger the gap, the smaller the amplitude (see comment).

We plot only the real part of $\psi(x)$. Also, we know that the particular shape of $\psi(x)$ depends on n. Here, we are just looking for how the amplitude & wavelength will behave for all n (please see the comment).



* Comment Q11 ~~a~~

we claimed ~~that~~ a larger E − V corresponds to smaller ~~wavelengths~~ amplitudes of $\psi(x)$.

our reasoning is

E − V larger $\overset{①}{\Rightarrow}$ the particle has more kinetic energy

② $\Rightarrow$ the particle spends less time in that region (since it moves faster)

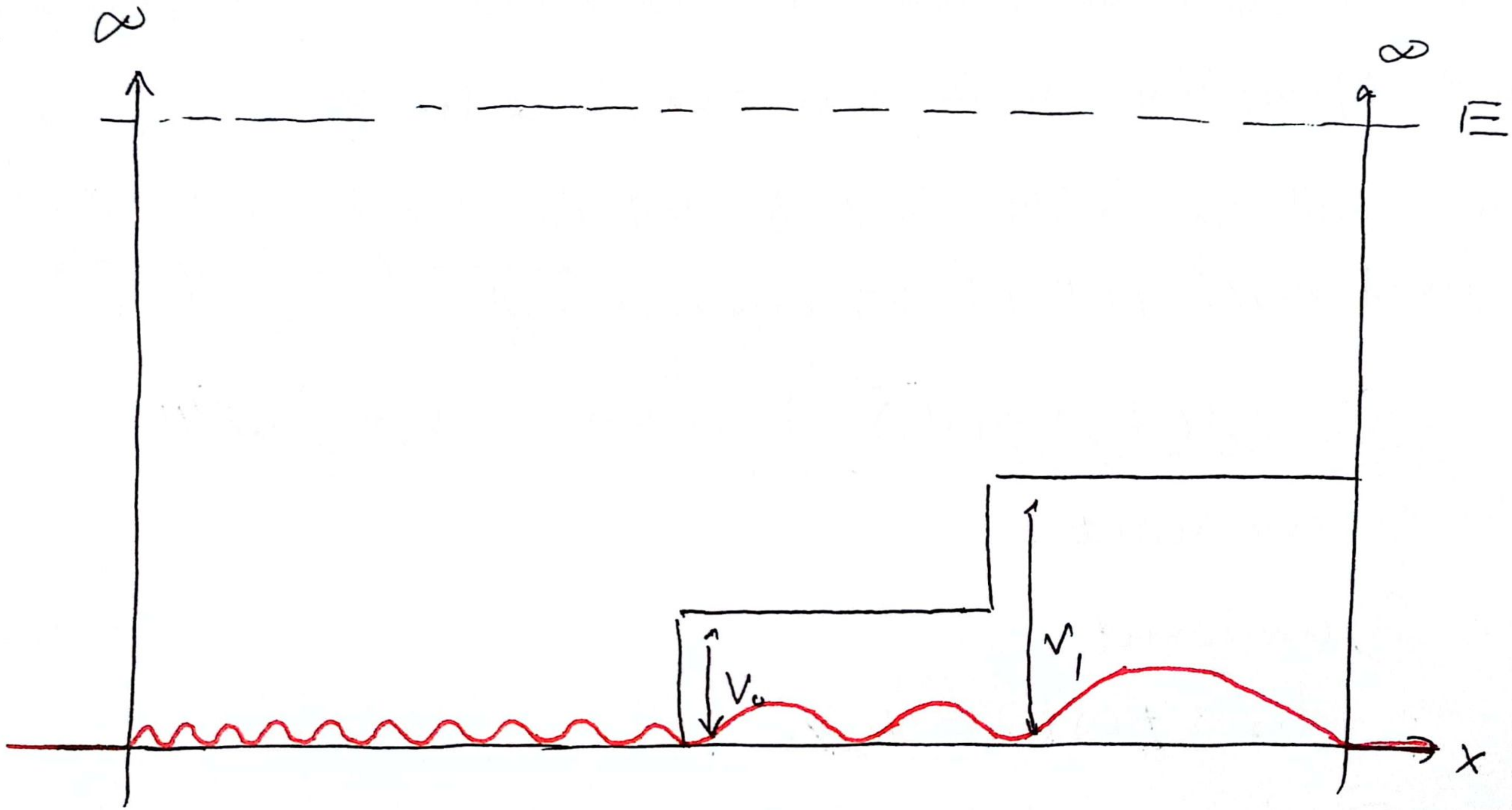
$\Rightarrow$ it is less likely to find the particle in that region

$\Rightarrow$ since $|\psi(x)|^2$ gives the probability density for position, it must be smaller in that region

$\Rightarrow$ $\psi(x)$ ~~has~~ has a smaller amplitude.

Note that this reasoning should be taken with a grain (or spoonfull of salt. Steps ① & ② ~~are~~ require us to think of a classical particle. There is no reason to expect this behavior to carry through to quantum mechanics. And indeed, it doesnot perfectly. But, using the correspondance principle, we can make the case that when n → larger values, our ~~plot~~ sketch for $\psi_n(x)$ with this reasoning will ~~get~~ better & better.

Q12



(13) We require that $\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1$. This just says that the probability of finding the particle between $+\infty$ & $-\infty$ is 1.

$$\int_{-\infty}^{0} (0)^2 \, dx + \int_{0}^{L} |A|^2 \cos^2\left(\frac{2n\pi}{L} x\right) dx + \int_{L}^{+\infty} (0)^2 \, dx = 1$$

$$\Rightarrow |A|^2 \int_{0}^{L} \cos^2\left(\frac{2n\pi}{L} x\right) dx = 1 .$$

$$\Rightarrow |A|^2 \int_{0}^{L} dx \left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{4n\pi x}{L}\right)\right] = 1$$

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$$\Rightarrow \frac{|A|^2}{2}\left[x - \frac{4n\pi}{L}\sin\left(\frac{4n\pi x}{L}\right)\right]_0^L = 1$$

$$\frac{|A|^2}{2}\left[L - \frac{4n\pi}{L}\sin(4n\pi) - 0\right] = 1$$

since $n \in \mathbb{Z}$, $\sin(4n\pi) = 0$

So

$$\frac{|A|^2 L}{\cancel{2}} = 1 \Rightarrow |A|^2 = \frac{2}{L}$$

$$\Rightarrow A = \sqrt{\frac{2}{L}}$$ upto a global phase.

variable function,