Assignment 3

1. We have been studying a Hamiltonian ($\hbar = 1$) that helps model interaction between spin-carrying particles:
$$H = -2\sum_{i<j} J_{ij}\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j. \tag{1}$$
This is often called the Heisenberg or exchange Hamiltonian. An important point to notice is that this model allows interactions between all particles, and at the outset we do not limit ourselves to, for example, nearest neighbour coupling. We begin with a small warm up exercise. Prove that the $\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j$ term can be written as
$$\hat{S}_{iz}\hat{S}_{jz} + \frac{1}{2}(\hat{S}_i^+\hat{S}_j^- + \hat{S}_i^-\hat{S}_j^+), \tag{2}$$
where $\hat{S}_i^+$ and $\hat{S}_i^-$ are the raising and lowering operators for the i^{th} spin site.

2. Consider a system of two particles that interact with the aforementioned Hamiltonian, and find its eigenenergies and eigenstates. Label the singlet and triplet eigen states. Now, you must prove that these eigenstates are also eigenstates of $\hat{\mathbf{S}}^2 = (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2$ with appropriate eigenvalues. Hint: you may look at Townsend's *Modern Approach to Quantum Mechanics* or Griffith's *Introduction to Quantum Mechanics*.

3. There are three spins each with $S = \frac{1}{2}$ placed in an equilateral triangle. They interact with each other through the Heisenberg Hamiltonian. (a) Find the eigenenergies of this configuration. (b) For the same configuration, find the eigenenergies for $S = 1$. (c) Let us consider the case where these spins interact through the Ising Hamiltonian:
$$H = -2J\sum_{i<j} \hat{S}_{iz}\hat{S}_{jz}. \tag{3}$$
What is the difference between the Ising model and the Heisenberg model? Find the eigenenergies associated with this Hamiltonian for $S = \frac{1}{2}$. Aside: the Ising model is an extremely useful model with applications from blackhole physics to quantum chaos.

4. Now consider a three dimensional system of N spins, all interacting with each other and each with spin S. Show that the state where all spins (M_S values) are aligned, $|\alpha\alpha\cdots\alpha\rangle$, is an eigenstate of the Heisenberg Hamiltonian. Note that $|\alpha\rangle \equiv |S, M_S\rangle$ and $M_S = S$. Consider interaction between next neighbours only and each spin is connected to d other spins. What is the ground state energy of this system?

5. In class, we transformed the Heisenberg Hamiltonian to a Hamiltonian that resembles one of a harmonic oscillator. We started by using the creation and annihilation operators for magnons, $\hat{b}_m$ and $\hat{b}_m^\dagger$. Then we carried out a Fourier transform to transform these operators, indexed by spatial points, to operators that are indexed by k, i.e., $\hat{a}_k$ and $\hat{a}_k^\dagger$. Assume that $\hat{b}_m$ and $\hat{b}_m^\dagger$ obey the following commutation relations:
$$[\hat{b}_m, \hat{b}_m^\dagger] = 1, \tag{4}$$

$$[\hat{b}_m, \hat{b}_m] = [\hat{b}^\dagger_m, \hat{b}^\dagger_m] = 0, \tag{5}$$

and the commutation relations vanish for all cases where indices do not match. Now, show that these commutation relations are also obeyed by the operators, $\hat{a}_k$ and $\hat{a}^\dagger_k$.

6. (a) Consider a ferromagnetic material in three dimensions, with each spin having d neighbours. Repeat the derivation to find the dispersion relation for magnons in three dimensions. (b) Now, we look at this dispersion relation in the small k regime. Using this expression, find the energy carried by magnons and the specific heat capacity for constant volume. Plot the temperature dependence of both quantities.

7. Prove the following identity:

$$\sum_j e^{i(k-k')ja} = N\delta_{k,k'}, \tag{6}$$

where j labels one of the N sites and a is the spacing between each site.

8. We study now the Bogoliubov transformation:

$$\begin{pmatrix} \hat{c}_{\mathbf{k}} \\ \hat{d}^\dagger_{-\mathbf{k}} \end{pmatrix} = \begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{pmatrix} \hat{\alpha}_{\mathbf{k}} \\ \hat{\beta}^\dagger_{-\mathbf{k}} \end{pmatrix}, \tag{7}$$

where p and q are from the reals. Assume that $[\hat{c}_{\mathbf{k}}, \hat{c}^\dagger_{\mathbf{k}}] = [\hat{d}_{\mathbf{k}}, \hat{d}^\dagger_{\mathbf{k}}] = 1$. Show that for the commutation relations,

$$[\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}^\dagger_{\mathbf{k}}] = [\hat{\beta}_{\mathbf{k}}, \hat{\beta}^\dagger_{\mathbf{k}}] = 1,$$

to be satisfied, we must set $p^2 - q^2 = 1$.