Assignment 4

1. The Holstein-Primakoff transformations are given by,

$$\hat{S}^{+} = \sqrt{2S}\sqrt{1 - \frac{\hat{a}^{\dagger}\hat{a}}{2S}}\hat{a}$$

$$\hat{S}^{-} = \hat{a}^{\dagger}\sqrt{2S}\sqrt{1 - \frac{\hat{a}^{\dagger}\hat{a}}{2S}},$$

where $[\hat{S}^{+}, \hat{S}^{-}] = 2\hat{S}_z$. Find $\hat{S}_z$ and check if it comes out as $S - \hat{a}^{\dagger}\hat{a}$.

2. In the low k limit ($k << \frac{1}{a}$), the dispersion for magnons in antiferromagnets is

$$\omega(k) \approx 4|J|Sak. \tag{1}$$

Find the total energy due to magnons in this limit and show that the heat capacity $C_v \propto T^3$.

3. Consider a solid with phonons (lattice vibrations) and magnons (spin waves) coupled together. The proposed Hamiltonian for such a system is given by

$$H = \sum_{\mathbf{k}} \left[\hbar\omega^{p}_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}}\hat{c}_{\mathbf{k}} + \hbar\omega^{m}_{\mathbf{k}}\hat{d}^{\dagger}_{\mathbf{k}}\hat{d}_{\mathbf{k}} + g(\hat{c}^{\dagger}_{\mathbf{k}}\hat{d}_{\mathbf{k}} + \hat{c}_{\mathbf{k}}\hat{d}^{\dagger}_{\mathbf{k}}) \right]. \tag{2}$$

Here $(\hat{c}_{\mathbf{k}}, \hat{c}^{\dagger}_{\mathbf{k}})$ and $(\hat{d}_{\mathbf{k}}, \hat{d}^{\dagger}_{\mathbf{k}})$ denote the operators for phonons and magnons respectively with frequencies $\omega^{p}_{\mathbf{k}}$ and $\omega^{m}_{\mathbf{k}}$, and g is the coupling between magnons and phonons. We wish to transform these aforementioned bosonic operators to bosonic operators in a rotated frame. To this end, we make the following transformation:

$$\begin{pmatrix} \hat{c}_{\mathbf{k}} \\ \hat{d}_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \cos(\theta_{\mathbf{k}}) & \sin(\theta_{\mathbf{k}}) \\ -\sin(\theta_{\mathbf{k}}) & \cos(\theta_{\mathbf{k}}) \end{pmatrix} \begin{pmatrix} \hat{\alpha}_{\mathbf{k}} \\ \hat{\beta}_{\mathbf{k}} \end{pmatrix}, \tag{3}$$

where $\theta_{\mathbf{k}}$ is real.

(a) Do $\hat{\alpha}_{\mathbf{k}}$ and $\hat{\beta}_{\mathbf{k}}$ obey the following commutation relation

$$[\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}^{\dagger}_{\mathbf{k}}] = [\hat{\beta}_{\mathbf{k}}, \hat{\beta}^{\dagger}_{\mathbf{k}}] = 1?$$

(b) Transform the Hamiltonian in (2) and determine when would the Hamiltonian have only terms that are like the number operator, $\hat{\alpha}^{\dagger}_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}}$ and $\hat{\beta}^{\dagger}_{\mathbf{k}}\hat{\beta}_{\mathbf{k}}$, with no cross terms, like $\hat{\alpha}^{\dagger}_{\mathbf{k}}\hat{\beta}_{\mathbf{k}}$ and $\hat{\alpha}_{\mathbf{k}}\hat{\beta}^{\dagger}_{\mathbf{k}}$.

(c) When $\omega^{p}_{\mathbf{k}} = \omega^{m}_{\mathbf{k}}$, what would be $\theta_{\mathbf{k}}$? For this condition, what would be the eigenenergies of the coupled magnon-phonon excitation? Furthermore, in this condition, how would the $(\hat{c}_{\mathbf{k}}, \hat{d}_{\mathbf{k}})$ and $(\hat{\alpha}_{\mathbf{k}}, \hat{\beta}_{\mathbf{k}})$ operators be related?

4. Consider an antiferromagnetic material subject to an external field B_0. Furthermore, the two sub-lattices experience a different internal field called the anistropy field B_a, about which we will learn later. The Hamiltonian can be written as

$$\hat{H} = |J| \sum_{<i,j>} \left[\hat{S}^{(A)}_{iz}\hat{S}^{(B)}_{jz} + \frac{1}{2}\left(\hat{S}^{(A)+}_{i}\hat{S}^{(B)-}_{j} + \hat{S}^{(A)-}_{i}\hat{S}^{(A)+}_{j}\right) - g\mu_B(B_0 + B_a)\left(\sum_i \hat{S}^{(A)}_{iz} + \sum_j \hat{S}^{(B)}_{jz}\right)\right], \tag{4}$$

where g is the Lande factor for spins in both the lattices.

(a) Use the Holstein-Primakoff transformations and Fourier transformations, ignoring higher order terms, to express (4) in terms of bosonic operators, $\hat{c}_{\mathbf{k}}$, $\hat{c}^{\dagger}_{\mathbf{k}}$ $\hat{d}_{\mathbf{k}}$, and $\hat{d}^{\dagger}_{\mathbf{k}}$.

(b) Apply the Bogoliubov transformation to express the Hamiltonian in (4) in terms of operators for bosonic Bogoliubov particles, $\hat{\alpha}^{\dagger}_{\mathbf{k}}$, $\hat{\alpha}_{\mathbf{k}}$, $\hat{\beta}^{\dagger}_{-\mathbf{k}}$, and $\hat{\beta}_{-\mathbf{k}}$.

(c) What should be the constraints on the Bogoliubov transformation to make the Hamiltonian quadratic, containing terms only of the kind $\hat{\alpha}^{\dagger}_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}}$ and $\hat{\beta}^{\dagger}_{-\mathbf{k}}\hat{\beta}_{-\mathbf{k}}$?

(d) Find the magnon frequencies for the $\hat{\alpha}$ and the $\hat{\beta}$ terms. Is the degeneracy lifted?