Midterm

Total time = 90 minutes

1. We have an electron (which is a fermion) of spin $\frac{1}{2}$. We define the angular momenta operators as

$$\hat{S}_i^+ = \hat{b}_{i\alpha}^\dagger \hat{b}_{i\beta}$$

$$\hat{S}_i^- = \hat{b}_{i\beta}^\dagger \hat{b}_{i\alpha}$$

where α denotes spin-up and β denotes spin-down. The operators $\hat{b}^\dagger$ and $\hat{b}$ are fermionic and satisfy,

$$\{\hat{b}_{i\alpha}^\dagger, \hat{b}_{i\beta}\} = \{\hat{b}_{i\alpha}, \hat{b}_{i\beta}^\dagger\} = \{\hat{b}_{i\alpha}^\dagger, \hat{b}_{i\beta}^\dagger\} = \{\hat{b}_{i\alpha}, \hat{b}_{i\beta}\} = 0$$

and

$$\{\hat{b}_{i\alpha}, \hat{b}_{i\alpha}^\dagger\} = \{\hat{b}_{i\beta}, \hat{b}_{i\beta}^\dagger\} = 1,$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A} = 1$ is the **anticommutator**. Anticommutator of operators at different sites are identically zero.

Express $\hat{S}_z$ as a function of number operator given that $[\hat{S}^+, \hat{S}^-] = 2\hat{S}_z$, where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the usual commutator. You may use the following identity,

$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}].$$

[10 Marks]

2. Consider a $J = S = 2$ particle. Suppose the energy in a field B is given by

$$E_J = -\alpha m_J B,$$

where $\alpha = -g_J \mu_B$.

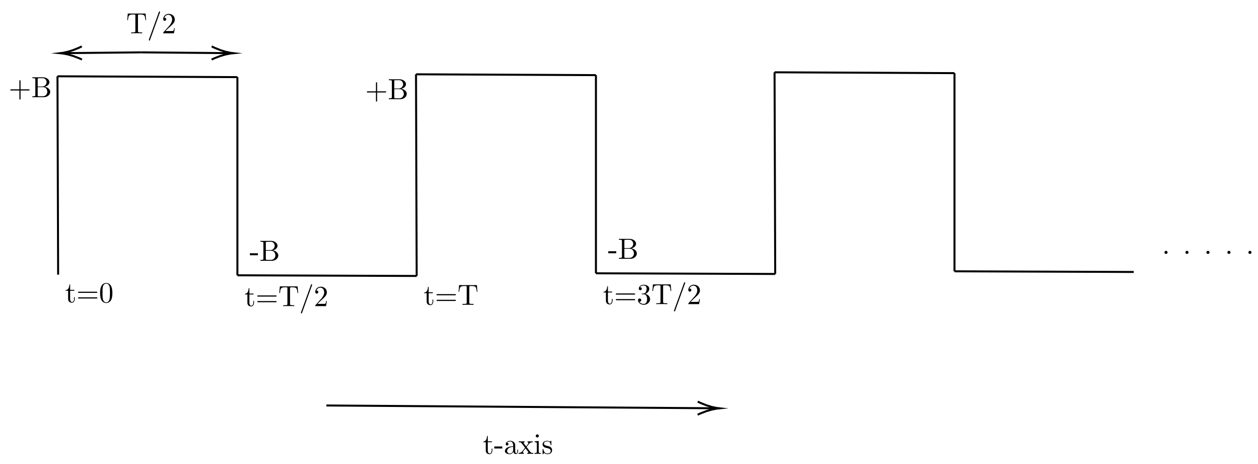
(a) Find the partition function Z for this particle.

(b) Find $\frac{1}{Z}\frac{\partial Z}{\partial x}$, where $x = \frac{\alpha B}{k_B T}$.

(c) Using the answer from part (b), find the magnetization (M), the saturation magnetization (M_S), and M/M_S. You should express your answers in terms of hyperbolic trigonometric functions.

[15 marks]

3. A magnetic moment $\vec{\mu}$ sees a pulsating magnetic field along the x-direction, shown in the figure below.



At $t = 0$, the magnetic moment is given by

$$\vec{\mu} = \mu_o \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mu_o \hat{z}.$$

Find the trajectory of the moment only during the first cycle of the pulse ($0 < t \leq \frac{T}{2}$ and $\frac{T}{2} \leq t < T$). Show your complete working. What ensures that the trajectory of the magnetic moment is periodic with a period of T?

[15 marks]

4. (a) Consider two spin-$\frac{1}{2}$ particles ferromagnetically coupled by the following interaction

$$\hat{H} = -2J\hat{S}_{1z}\hat{S}_{2z},$$

where $J > 0$. Find the two energy states, lower and higher, and their energies.

(b) What is the partition function for this pair of spins?

(c) What is the probability that the spin pair is in the lower energy state?

(d) Now extend this pair to a linear chain of N spins or $(N-1)$ pairs. The Hamiltonian now becomes

$$\hat{H} = -2J\sum_{m=1}^{N-1} \hat{S}_{m,z}\hat{S}_{m+1,z} = -2J\sum_{m=1}^{N-1} \hat{Q}_{m,z},$$

where $\hat{Q}_{m,z} = \hat{S}_{m,z}\hat{S}_{m+1,z}$. Using your answer to part (b), what is the partition function for this complete linear chain?

(e) Using your answer to part (c), what is the probability that F consecutive spins are parallel to one another? What is this probability at low temperatures? (Note that $e^{a \ln(x)} = x^a$.)

[20 marks]