Homework 1

Problem 1

Calculate the energy of the n'th excited state to second order and the wave function to first order for a one-dimensional box potential of length L, with infinitely high walls at $x = 0$ and $x = L$, which is modified at the bottom by the following perturbations with $V_o \ll 1$:

(a)

$$V_p(x) = \begin{cases} 0 & 0 < x < L/2, \\ V_o & L/2 \leq x < L \end{cases}$$

(b) Suppose we put a delta-function bump in the center of the infinite well

$$V_p(x) = V_o \; \delta(x - L/4)$$

where V_o is a constant.

Problem 2

Consider an one-dimensional harmonic oscillator described by the potential energy

$$V(x) = \frac{m}{2}\omega_o^2 x^2 + bx^3$$

compute the energy eigenvalues to second order and eigenstates to first order. The cubic term represents a small perturbation, modifying the idealized harmonic oscillator.

Problem 3

(Sakurai Q.5.3) Consider a particle in a two-dimensional potential:

$$V_0 = \begin{cases} 0, & \text{for } 0 \leq x \leq L, \; 0 \leq y \leq L, \\ \infty, & \text{otherwise.} \end{cases}$$

Write the energy eigenfunctions for the ground state and the first excited state.

We now add a time-independent perturbation of the form:

$$V_p = \begin{cases} \lambda \, xy, & \text{for } 0 \leq x \leq L, \; 0 \leq y \leq L, \\ 0, & \text{otherwise.} \end{cases}$$

Obtain the zero'th-order energy eigenfunctions and the first-order energy shifts for the ground state and the first excited state.

Problem 4

Consider a system whose Hamiltonian is given by:

$$\hat{H} = E_0 \begin{pmatrix} 5 & 3\lambda & 0 & 0 \\ 3\lambda & 5 & 0 & 0 \\ 0 & 0 & 8 & -\lambda \\ 0 & 0 & -\lambda & 5 \end{pmatrix},$$

where $\lambda \ll 1$ represents a small perturbation.

(a) By decomposing this Hamiltonian into $\hat{H} = \hat{H}_o + \hat{H}_p$, find the eigenvalues and eigenstates of the unperturbed Hamiltonian H_o.

(b) Diagonalize $\hat{H}$ to find the exact eigenvalues of $\hat{H}$; expand each eigenvalue to the second power of λ.

(c) Use perturbation theory to find the approximate eigenenergies of $\hat{H}$. Compare these with the exact values obtained in (b).

Problem 5

(Zetilli Q.9.11) Consider an isotropic three-dimensional harmonic oscillator.

(a) Find the energy of the first excited state and the different states corresponding to this energy.

(b) If we now subject this oscillator to a perturbation $\hat{V}_p(x, y) = -\lambda \hat{x}\hat{y}$, where λ is a small real number, find the energy of the first excited state to first-order degenerate time-independent perturbation theory.

Hint: You may use:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a}_x + \hat{a}_x^\dagger\right), \quad y = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a}_y + \hat{a}_y^\dagger\right), \quad z = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a}_z + \hat{a}_z^\dagger\right),$$

where:

$$\hat{a}_x|n_x\rangle = \sqrt{n_x}|n_x - 1\rangle, \quad \hat{a}_x^\dagger|n_x\rangle = \sqrt{n_x + 1}|n_x + 1\rangle.$$

Problem 6

Write down the angular momentum operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ represented in the joint eigenbasis of the $\hat{S}_z$ and $\hat{S}^2$ operators. Consider a $s = 3/2$ particle.