Quantum Optics: HW1 by Muhammad Sabieh Anwar

Please submit before 10 am, Thursday, 6 February 2025

1. Consider N atoms inside an open cavity exposed to radiation of energy density $u(\omega)$. The radiation enters the cavity at time $t = 0$. Answer the following questions. Each atom has two levels and all of them are initially in the ground state.

   (a) Using Einstein's rate equations, find and plot the fraction of atoms in the excited state, N_2, as a function of time.

   (b) Plot the steady state population N_2, as a function of the energy density $u(\omega)$.

   (c) What is the steady state population of N_2 and how does the population behave at short times?

   (d) Answer question (a) for the population N_1.

   (e) What is the maximum achievable population of N_2 and what energy density is required for that?

2. The energy density in a certain volume of space is $u(\omega)$. Show that the intensity (energy per unit area per unit time) received on a surface is $I = \alpha c u$ where c is the speed of light. The factor $\alpha = 1$ for a collimated beam of radiation as in a laser and $\alpha = 1/4$ for a source radiating isotropically in all directions. Using Planck's radiation formula, and the latter sesult, show that the energy radiated per unit time per unit area from a source is $R = \sigma T^4$. This is called Stefan-Boltzmann's law. In the process determine the value of the constant σ. You may have to

use the integral

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}. \tag{1}$$

3. Using Einstein's description of absorption and emission, and Planck's radiation formula, determine the temperature at which spontaneous and stimulated emission rates become equalized. What is this temperature for the following?

   (a) microwave radiation at 10 GHz

   (b) optical radiation at 600 nm

   (c) X-rays of energy 10 KeV

4. Express the total emission rate from a two-level system, inclusive of spontaneous and stimulated emission in terms of the number of photons $\langle n_\omega \rangle$ inside the system. Under what conditions would stimulated emission dominate and is this feasible for a two-level system?

5. The diagram (Fig.1) shows a Lorentzian lineshape $g(\omega)$ with a peak at ω_o and FWHM=Δ. Identify the FWHM and centre frequency on this figure. We need to keep $\int d\omega\, g(\omega) = 1$. What is the height of the peak? Show that this is the Fourier transform a decaying sinusiod and relate Δ to the decay rate.

6. Here are some properties of gaseous mercury atoms in a discharge lamp: mass= 200 atomic units, $T = 200°\text{C}$, Einstein coefficient $A = 5 \times 10^7\ \text{s}^{-1}$, atomic radius= 0.17 nm.

   (a) Estimate the natural linewidth.

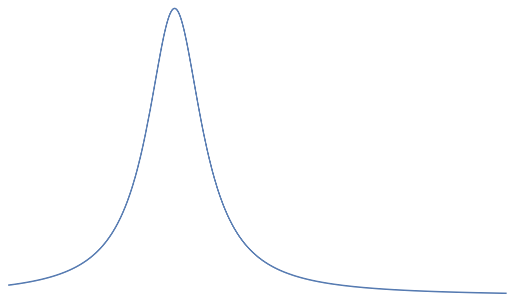

Figure 1: Lorentzian spectral lineshape.

(b) Show the complete working for determining the pressure broadened linewidth at a pressure P for an arbitrary gas.

(c) Estimate the pressure broadened linewidth given a pressure $P =$ 1 atm, and $P = 10^{-3}$ atm.

(d) At what pressure would the pressure broadened linewdith be a tad smaller than the naturally broadened linewidth?

(e) Given the linewidth in terms of frequency $\Delta\omega$ at ω_o, find the corresponding linewdith in terms of wavelengths $\Delta\lambda$, at λ_o.

7. In class, we discussed the Lorentz oscillator model in one dimension. Let's upgrade this to three-diemnsional space. The equation of motion, with the usual meanings of the symbols, becomes:

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega_o^2\mathbf{r} - m\gamma\frac{d\mathbf{r}}{dt} - \frac{e\mathbf{E}_o}{m}e^{-i\omega t}, \tag{2}$$

where $\mathbf{r}$ is the position vector. Bold fonts represent three-dimensional vectors.

(a) Propose a solution $\mathbf{r}$, and verify by inserting this ansatz into the equation of motion.

(b) Find the electric dipole vector $\mathbf{p} = -e\mathbf{r}$.

(c) Find the polarization $\mathbf{P} = N\mathbf{p}$ where N is the total number of oscillators.

(d) Using $P + \varepsilon_o(\varepsilon_r - 1)\mathbf{E}$, find the refractive index $n = \sqrt{\varepsilon_r}$. Is the refractive index complex? Note that $\varepsilon_{(o,r)}$ represents the vacuum (relative) permittivity.

(e) What do the real and imaginary parts of the refractive index represent?

(f) Plot the real and imaginary parts *only* in the vicinity of $\omega \approx \omega_o$.