PHY 312 **Due by:**
February 14, 2025 February 28, 11 am

Homework 2

Problem 1

Consider a hydrogen atom in the quantum state characterized by the quantum numbers $n = 3$, $l = 1$, and $m = -1$. Answer the following questions:

(a) Compute the probability of finding the electron in the angular region $30^\circ \leq \theta \leq 60^\circ$, integrating over all values of ϕ.

(b) Plot the radial wavefunction $R_{31}(r)$ and the corresponding radial probability density $P_r(r) = r^2|R_{31}(r)|^2$. Determine the number of radial nodes in $R_{31}(r)$.

(c) Find the value of r at which the radial probability density $P_r(r)$ is maximized. This corresponds to the most probable radial distance of the electron in this quantum state.

Problem 2

Consider the function

$$\Psi(r, \theta, \phi) = A(x + iy)e^{-r/2a_0},$$

where a_0 is the Bohr radius and A is a real constant.

(a) Express $\Psi(r, \theta, \phi)$ in terms of $R_{nl}(r)Y_{lm}(\theta, \phi)$ and determine the quantum numbers n, l, m.

(b) Plot the function $R_{nl}(r)$ and $r^2|R_{nl}(r)|^2$ using the values of n, l obtained in part (a).

(c) Find the constant A such that $\Psi(r, \theta, \phi)$ is normalized.

(d) Find the mean value of r and the most probable value of r in this state.

Problem 3

(Zetilli Q.6.22) Calculate Δr and Δp_r with respect to the state

$$\Psi_{210}(r, \theta, \phi) = \frac{1}{\sqrt{6}} \left(\frac{Z}{a_0}\right)^{3/2} Zre^{-Zr/2a_0} Y_{10}(\theta, \phi),$$

and verify that $\Delta r \Delta p_r$ satisfies the Heisenberg uncertainty principle, where $\Delta r = \sqrt{\langle \hat{r}^2 \rangle - \langle \hat{r} \rangle^2}$ and $\Delta p_r = \sqrt{\langle \hat{p_r}^2 \rangle - \langle \hat{p_r} \rangle^2}$.
Note: Z=1 for Hydrogen atom.

Problem 4

A hydrogen atom starts out in the following linear combination of the stationary states $n = 2, l = 1, m = 1$ and $n = 2, l = 1, m = -1$:

$$\Psi(\vec{r}, 0) = \frac{1}{\sqrt{2}} \left(\psi_{211} + \psi_{21-1}\right).$$

(a) Plot the spherical harmonics of the stationary states used in this question i-e $(Y_{11}(\theta, \phi), Y_{1-1}(\theta, \phi))$.

(b) Construct $\Psi(\vec{r}, t)$. Simplify it as much as you can.

(c) Find the expectation value of the potential energy, $\langle V \rangle$. (Does it depend on t?) Give both the formula and the actual number, in electron volts.

Problem 5

(Griffiths Q.4.45) What is the probability that an electron in the ground state of hydrogen will be found *inside the nucleus*?

(a) First, calculate the **exact** answer, assuming the wave function is correct all the way down to $r = 0$. Let b be the radius of the nucleus.

(b) Expand your result as a power series in the small number $\varepsilon = 2b/a$, and show that the lowest-order term is the cubic:

$$P \approx \frac{4}{3}\left(\frac{b}{a}\right)^3.$$

This should be a suitable approximation, provided that $b \ll a$ (which it is).

(c) Alternatively, we might assume that $\psi(r)$ is essentially constant over the (tiny) volume of the nucleus; so that

$$P \approx \frac{4}{3}\pi b^3 |\psi(0)|^2.$$

Check that you get the same answer this way.

(d) Use $b \approx 10^{-15}$ m and $a \approx 0.5 \times 10^{-10}$ m to get a numerical estimate for P. Roughly speaking, this represents the "fraction of its time that the electron spends inside the nucleus."

Problem 6

At time $t = 0$, a hydrogen atom is in the superposition state:

$$\psi(r, 0) = \frac{4}{(2a_0)^{3/2}} \left[e^{-r/a_0} + A \frac{r}{a_0} e^{-r/2a_0} (-iY_1^1 + Y_1^{-1} + \sqrt{7} Y_1^0) \right].$$

(a) Calculate the value of the normalization constant A.

(b) What is the probability density $P_r(r)$ that the electron is found in the shell of thickness dr about the proton at the radius r?

(c) At what value of r is $P_r(r)$ maximum?

(d) Given the initial state $\psi(r, 0)$, what is $\psi(r, t)$?

(e) What is the expectation of the "spherical energy operator," $\langle H_S \rangle$, where

$$\hat{H}_S \equiv \hat{H} - \frac{\hat{p}_r^2}{2\mu}.$$

(f) What is the lowest value of energy that measurement will find at $t = 0$? (Lowest means the negative value farthest removed from zero.)