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* Quantum Optics
* Assignment #1

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1) a) $$\frac{dN_1}{dt} = \underbrace{-B_{12}U(\omega)N_1}_{\text{absorption}} + \underbrace{AN_2}_{\text{spontaneous emission}} + \underbrace{B_{21}U(\omega)N_2}_{\text{stimulated emission}}$$

$$N = N_1 + N_2 \quad \text{and} \quad \frac{dN}{dt} = 0$$

$$\Rightarrow \quad \frac{dN_2}{dt} = -\frac{dN_1}{dt}$$

$$\frac{dN_2}{dt} = B_{12}U(\omega)N_1 - AN_2 - B_{21}U(\omega)N_2$$

At thermal equilibrium, we have the following:

$$B_{12}N_1U(\omega) = AN_2 + B_{21}U(\omega)N_2 \quad \because \frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$$

$$\Rightarrow U(\omega) = \frac{AN_2}{B_{12}N_1 - B_{21}N_2} = \frac{A}{B_{12}\left(\frac{N_1}{N_2}\right) - B_{21}}$$

By Boltzmann's law, we have:

$$\frac{N_2}{N_1} = e^{-\frac{\hbar\omega}{k_BT}}$$

Energy spectrum of a black-body source is given by:

$$U(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} = \frac{\hbar\omega^3}{\pi^2 c^3} \cdot \frac{1}{\frac{N_1}{N_2} - 1}$$

Putting all these equations together, we have: $\underline{\underline{B_{12} = B_{21} = B}}$

$$\boxed{\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3}}$$

$= Bu(\omega)(N_1) - AN_2 - Bu(\omega)N_2$

$$\frac{dN_2}{dt} = Bu(\omega)(N - N_2) - AN_2 - Bu(\omega)N_2$$

$$= -(2Bu(\omega) + A)N_2 + Bu(\omega)N$$

Please show the derivation.

$$\Rightarrow N_2(t) = \frac{NBu(\omega)}{2Bu(\omega) + A}\left(1 - e^{-(2Bu(\omega) + A)t}\right)$$

$$U_s(\omega) := \frac{A}{B}$$

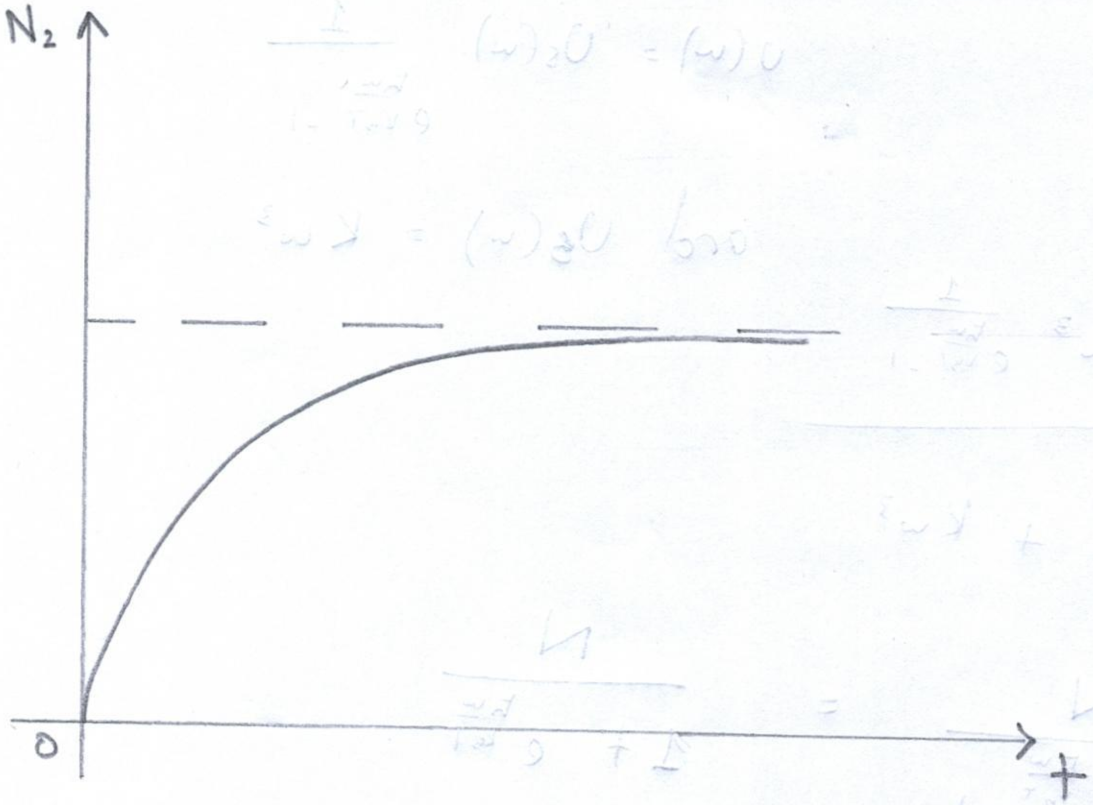
$$N_2(0) = 0$$

$$N_2(\infty) = \frac{NBu(\omega)}{2Bu(\omega) + A} \Rightarrow \boxed{N_2(\infty) = \frac{Nu(\omega)}{2u(\omega) + U_s(\omega)}}$$

Steady state population.

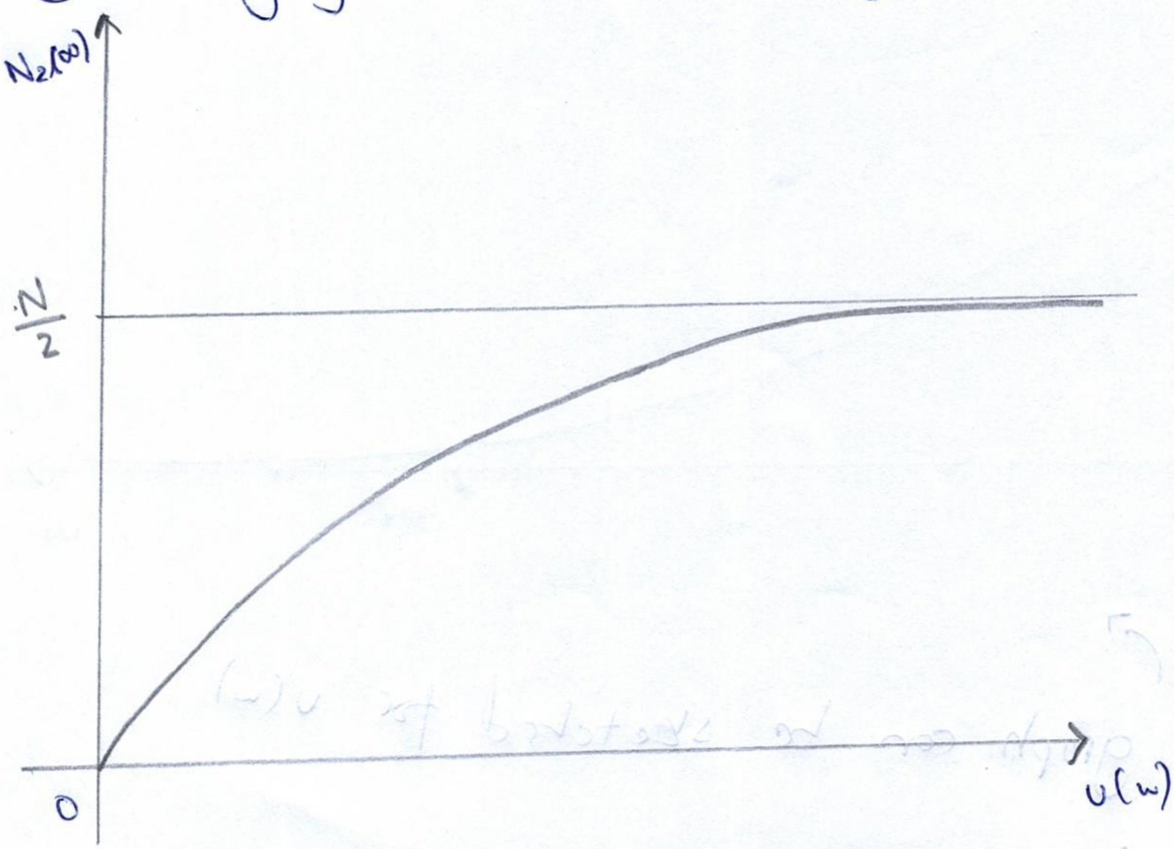
→ This cannot go beyond $\frac{1}{2}$, even if $u(\omega) \to \infty$.

→ For $u(\omega) = u_s(\omega)$; $\frac{N_2(\infty)}{N} = \frac{1}{3}$.



b) $$N_2(\infty) = \frac{N\upsilon(w)}{2\upsilon(w) + \upsilon_s(w)}$$

1 Changing $\upsilon(w)$ without changing $\upsilon_s(w)$



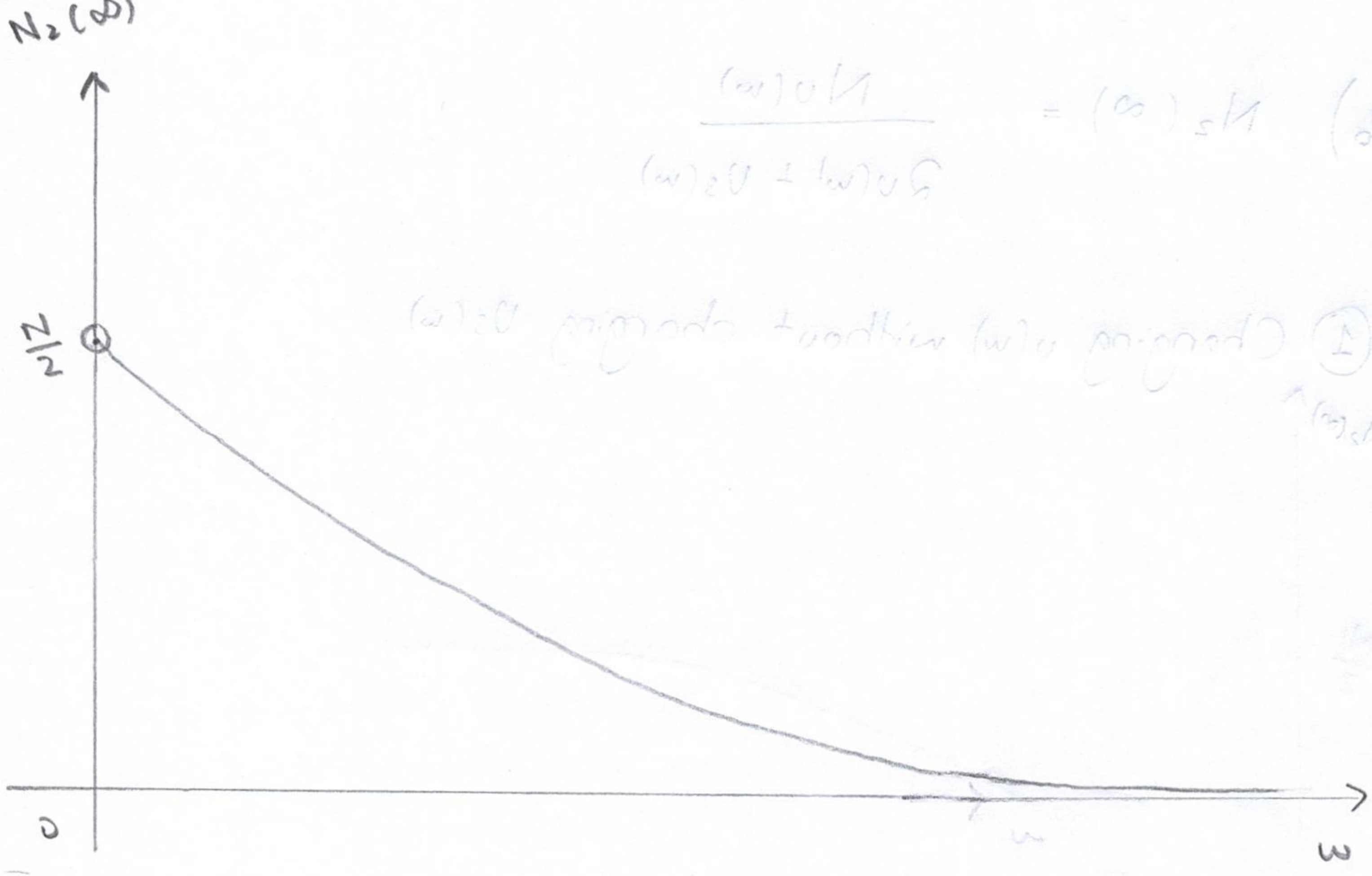
② Changing ω

$$\upsilon(\omega) = \upsilon_s(\omega)\,\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}$$

and $\upsilon_s(\omega) = K\omega^3$

$$N_2(\infty) = \frac{N K\omega^3 \frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}}{2K\omega^3 \frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1} + K\omega^3}$$

$$N_2(\infty) = \frac{N}{2 + e^{\frac{\hbar\omega}{k_BT}}-1} = \frac{N}{1 + e^{\frac{\hbar\omega}{k_BT}}}$$



③ A similar graph can be sketched for $\upsilon(\omega)$ against $\frac{1}{T}$.

c) $$N_2(t) = \frac{NBu(w)}{2Bu(w) + A}\left(1 - e^{-(2Bu(w)+A)t}\right)$$

t is small

$$N_2(t) = \frac{NBu(w)}{2Bu(w) + A}\left(1 - \left(1 - (2Bu(w)+A)t\right)\right)$$

$$= [NBu(w)]\, t$$

The population increases linearly with time.

t = ∞

$$N_2(\infty) = \frac{NBu(w)}{2Bu(w) + A} = \frac{NBu(w)}{2Bu(w) + Bu_s(w)}$$

$$= \frac{Nu(w)}{2u(w) + u_s(w)}$$

Steady-state population

d) $N_1(t) = N - N_2(t)$

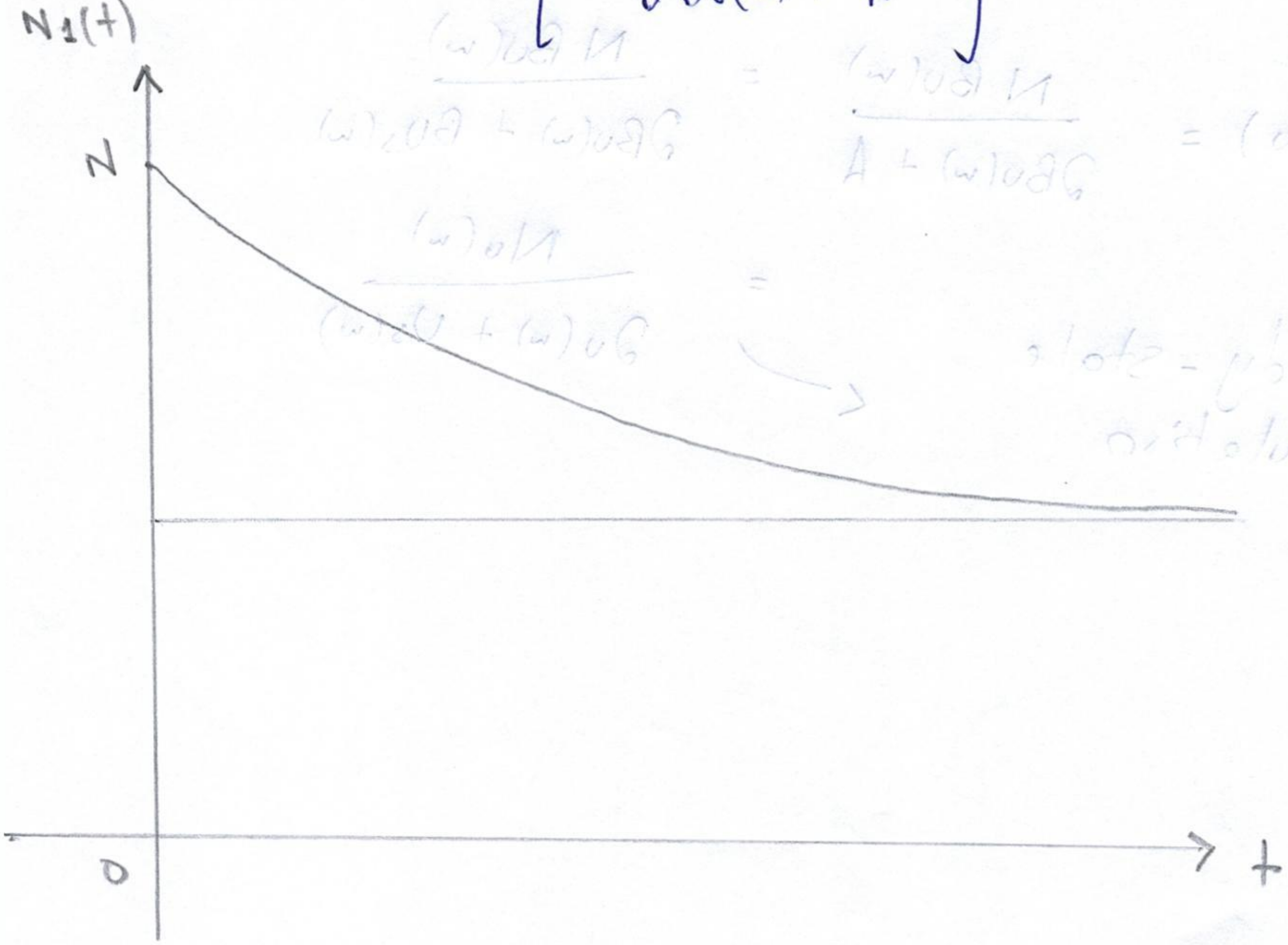
$$= N + \frac{NBu(w)}{2Bu(w)+A}\left(e^{-(2Bu(w)+A)t} - 1\right)$$

$N_1(0) = N$

$$N_1(\infty) = N - \left(\frac{NBu(w)}{2Bu(w)+A}\right)$$

$$= N\left[\frac{2Bu(w)+A - Bu(w)}{2Bu(w)+A}\right]$$

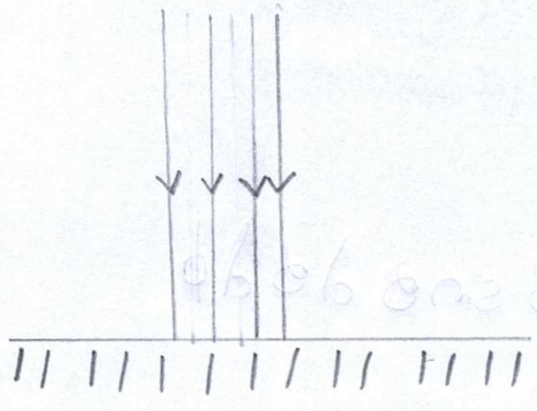
$$= N\left[\frac{A + Bu(w)}{2Bu(w)+A}\right]$$



e) $\frac{N}{2}$ is the maximum achievable population of N_2; $N_2(\infty) = \frac{N u(\omega)}{2u(\omega) + U_s(\omega)}$; the

energy density $u(\omega)$ require for $N_2(\infty) = \frac{N}{2}$ is $u(\omega) = \infty$ ($U_s(\omega)$ is finite)

2) ① Energy received on a surface from a collimated beam of radiation



$dE = (c)(dt)(dA)(u)$

$c.dt$ = distance traveled by the light

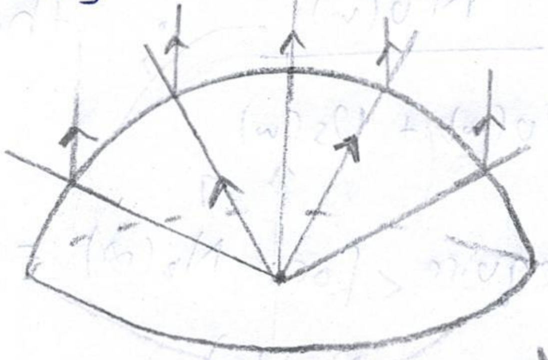
$(c.dt).(dA)$ = volume of light falling on surface

$(cdt\, dA).u$ = energy falling on surface

$$\frac{dE}{dt.dA} = cu \Rightarrow \boxed{I = cu}$$

$(\alpha = 1)$

② Energy received ~~fr~~ from a source radiating isotropically in all directions



$$\frac{dE}{dA} = \overbrace{(c\cos\theta)(dt)}^{\text{distance traveled by light}} \underbrace{\left(\frac{u}{4\pi}\right)}_{\text{energy density per solid angle}} \overbrace{d\Omega}^{\text{differential solid angle}}$$

$$I = \frac{cu}{4\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{2\pi} \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= \frac{cu \cdot 2\pi}{4\pi} \cdot \frac{1}{2} \cdot \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= \frac{cu}{4} \cdot \left| -\frac{1}{2}\cos 2\theta \right|_0^{\pi/2}$$

$$= \frac{cu}{4}\left[\frac{1}{2} + \frac{1}{2}\right] = \boxed{\frac{1}{4} cu}$$

$$u(\omega) = \frac{\hbar\omega^3}{c^3\pi^2} \cdot \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}$$

$$R = \frac{1}{4} \cdot c \cdot \frac{\hbar}{c^3\pi^2} \int_0^\infty \frac{\omega^3}{e^{\frac{\hbar\omega}{k_B T}} - 1}\, d\omega$$

$$= \frac{\hbar}{4c^2\pi^2} \cdot \left(\frac{k_B T}{\hbar}\right)^3 \int_0^\infty \frac{\left(\frac{\hbar\omega}{k_B T}\right)^3}{e^{\frac{\hbar\omega}{k_B T}} - 1}\, d\omega$$

$$x := \frac{\hbar\omega}{k_B T}$$

$$dx = \frac{\hbar}{k_B T}\, d\omega$$

$$= \frac{\hbar}{4c^2\pi^2} \left(\frac{k_B T}{\hbar}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}\, dx$$

$$= \frac{\hbar}{4c^2\pi^2} \cdot \frac{k_B^4 \cdot T^4}{\hbar^4} \cdot \frac{\pi^4}{15}$$

$$= \frac{k_B^4 \cdot T^4 \cdot \pi^2}{4c^2 \cdot \hbar^3 \cdot 15} = \frac{k_B^4 T^4 \cdot \pi^2}{4c^2 \cdot h^3 \cdot 15} \cdot 8\pi^3$$

$$= \left(\frac{2k_B^4\pi^5}{15c^2h^3}\right) T^4 \Rightarrow \boxed{\sigma = \frac{2k_B^4\pi^5}{15c^2h^3}}$$

$$u(\omega) = \frac{\hbar\omega^3}{c^2\pi^2} \cdot \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}$$

$$\sigma = \frac{1}{4} \cdot c \cdot \frac{\hbar}{c^2\pi^2} \int_0^\infty \frac{\omega^3}{e^{\frac{\hbar\omega}{kT}} - 1}\, d\omega$$

$$= \frac{\hbar}{4c^2\pi^2} \cdot \left(\frac{kT}{\hbar}\right)^4 \int_0^\infty \frac{\left(\frac{\hbar\omega}{kT}\right)^3}{e^{\frac{\hbar\omega}{kT}} - 1}\, d\frac{\hbar\omega}{kT}$$

$$x := \frac{\hbar\omega}{kT}$$

$$dx = \frac{\hbar}{kT}\, d\omega$$

$$= \frac{\hbar}{4c^2\pi^2} \left(\frac{kT}{\hbar}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}\, dx$$

$$= \frac{\hbar}{4c^2\pi^2} \cdot \frac{k^4 T^4}{\hbar^4} \cdot \frac{\pi^4}{15}$$

$$= \frac{k^4 \cdot T^4 \cdot \pi^2}{4c^2 \cdot \hbar^3 \cdot 15} = \frac{\pi^2 k^4}{60 c^2 \hbar^3} \cdot T^4$$

$$= \left(\frac{\pi^2 k^4}{60 c^2 \hbar^3}\right) T^4 \implies \boxed{\sigma = \frac{\pi^2 k^4}{60 c^2 \hbar^3}}$$

3) from Q1, we have that

$$\boxed{B N_1 u(\omega) = A N_2 + B u(\omega) N_2} \Rightarrow \boxed{\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3}}$$

* Thermal Equilibrium assumed

spontaneous emission = stimulated emission

$$A N_2 = B u(\omega) N_2$$

$$\boxed{u(\omega) = \frac{A}{B}} \longrightarrow$$ this is also called $u_s(\omega)$ saturation energy density. (when spontaneous emission rate = stimulated emission rate = absorption rate).

$$\Rightarrow \quad u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}$$

$$\frac{\hbar\omega^3}{c^3\pi^2}\left(\frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}\right) = \frac{\hbar\omega^3}{\pi^2 c^3}$$

$$\Rightarrow \quad e^{\frac{\hbar\omega}{kT}} - 1 = 1$$

$$e^{\frac{\hbar\omega}{kT}} = 2$$

$$\frac{\hbar\omega}{kT} = \ln 2$$

$$\Rightarrow \quad \boxed{T = \frac{\hbar\omega}{k_B \ln 2}}$$

a) $T \approx \boxed{0.690\,K}$

b) $\lambda = 600nm$

$f = 5\times10^{14}$ Hz

$\boxed{T \approx 3469K}$

c) $hf = 10keV$

$f = 2.413\times10^{18}$ Hz

$T \approx \boxed{1.67\times10^{8}\,K}$

4) Spontaneous emission $= AN_2$

Stimulated emission $= B\,u(\omega)\,N_2$

$$= B\,\frac{\hbar\omega^3}{c^3\pi^2}\cdot\underbrace{\left(\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}\right)}_{\text{average \# of photons}}\cdot N_2$$

$$= (B)\,\frac{\hbar\omega^3}{c^3\pi^2}\cdot\langle n_\omega\rangle\cdot N_2$$

Total emission rate $= AN_2 + B\cdot\frac{\hbar\omega^3}{c^3\pi^2}\langle n_\omega\rangle\cdot N_2$

$$= (A)(N_2)\left(1+\frac{B}{A}\cdot\frac{\hbar\omega^3}{c^3\pi^2}\langle n_\omega\rangle\right)$$

$\hookrightarrow = c^3\pi^2/\hbar\omega^3$

$$= \boxed{(A\,N_2)(1+\langle n_\omega\rangle)}$$

Stimulated emission $>$ Spontaneous emission

$$B\cdot\frac{\hbar\omega^3}{c^3\pi^2}\langle n_\omega\rangle \cancel{N_2} > A\cancel{N_2}$$

$\frac{A}{B}$ cancels $\frac{\hbar\omega^3}{c^3\pi^2}$

$\Rightarrow \langle n_\omega\rangle > 1$ $\rightarrow$ $\langle n_\omega\rangle$ must be $\gg 1$ for stimulated emission to dominate!

5) $x(t) = x_0 e^{-\frac{\gamma t}{2}} \cos(w_0 t + \phi_0)$

The $\vec{E}$ field emitted by the e^- is given as:

$$\vec{E}(t) = \begin{cases} 0 & t < 0 \\ E_0 \cos w_0 t \, e^{-\frac{\gamma t}{2}} & t \geq 0 \end{cases}$$

$$\mathcal{E}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(t) e^{-iwt} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} E_0 e^{-\frac{\gamma t}{2}} \cdot e^{iw_0 t} \cdot e^{-iwt} \, dt$$

$$= \frac{E_0}{\sqrt{2\pi}} \left| \frac{1}{-\frac{\gamma}{2} + i(w_0 - w)} \cdot e^{\left[-\frac{\gamma}{2} + i(w_0 - w)\right] t} \right|_{t=0}^{\infty}$$

$$= \frac{E_0}{\sqrt{2\pi} \left(\frac{\gamma}{2} + i(w - w_0) \right)}$$

$$|E(w)|^2 = \frac{E_0^2}{2\pi} \cdot \left(\frac{1}{\frac{\gamma}{2} + i(w - w_0)} \right) \left(\frac{1}{\frac{\gamma}{2} - i(w - w_0)} \right)$$

$$= \left(\frac{E_0^2}{2\pi} \right) \left(\frac{1}{\left(\frac{\gamma}{2}\right)^2 + (w - w_0)^2} \right)$$

$$I(\omega) \propto \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2} \Longrightarrow \frac{1}{(\omega-\omega_0)^2 + \left(\frac{1}{2\tau}\right)^2}$$

$$\gamma := \frac{1}{\tau}$$

$$\underbrace{\Delta\omega}_{\text{FWHM}} := \frac{1}{\tau} \qquad \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\Delta\omega}{2}\right)^2}$$

$$N \int_{-\infty}^{\infty} \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\Delta\omega}{2}\right)^2} \, d\omega = 1$$

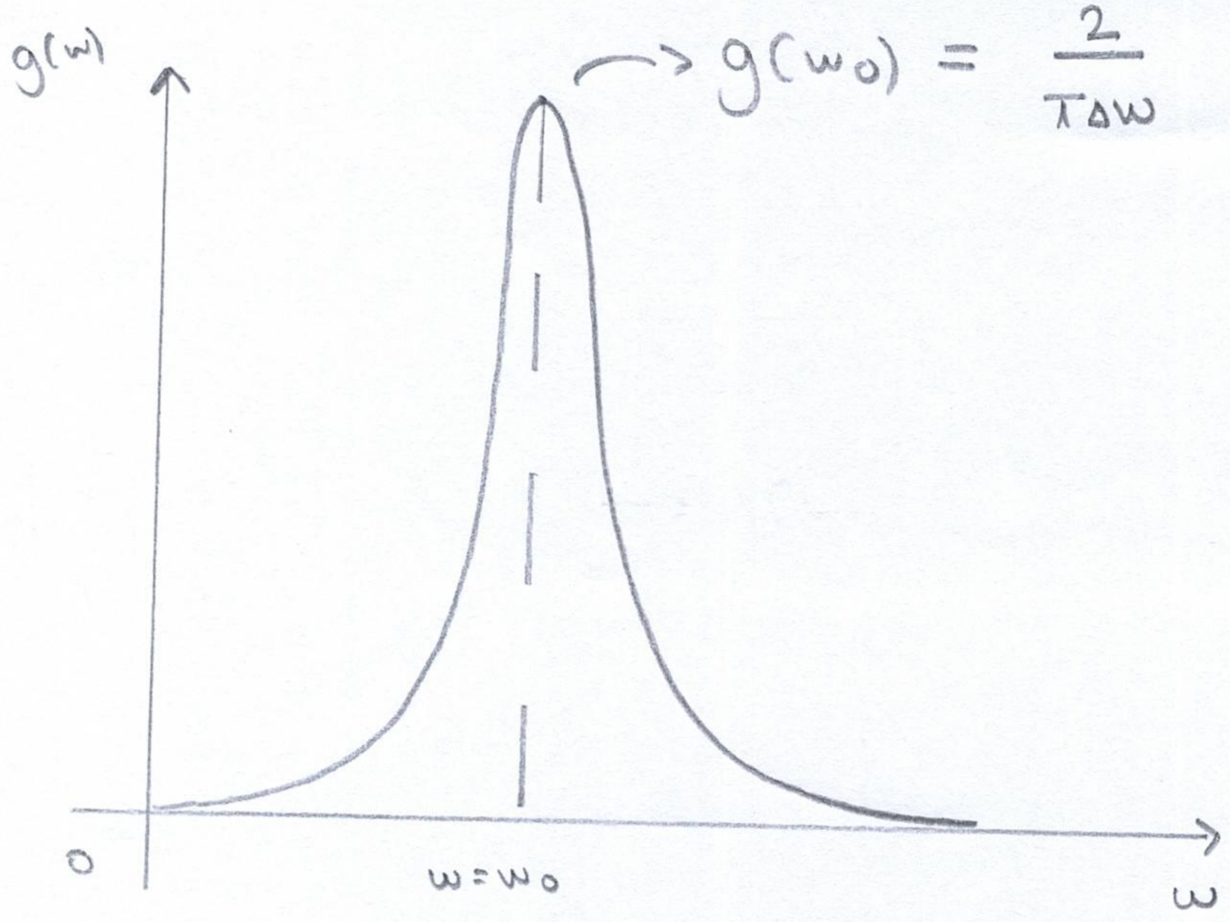
using $\int_{-\infty}^{\infty} \frac{du}{x^2+a^2} = \frac{\pi}{a}$

$$N \cdot \frac{\pi}{\Delta\omega} \cdot 2 = 1$$

$$\Longrightarrow N = \frac{\Delta\omega}{2\pi}$$

Lorentzian Lineshape

$$\boxed{g(\omega) = \frac{\Delta\omega}{2\pi} \cdot \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\Delta\omega}{2}\right)^2}}$$



$$\frac{1}{\pi \Delta w} = \frac{\Delta w}{2\pi} \cdot \frac{1}{(w-w_0)^2 + \left(\frac{\Delta w}{2}\right)^2}$$

$$\frac{2}{(\Delta w)^2} = \frac{1}{(w-w_0)^2 + \left(\frac{\Delta w}{2}\right)^2}$$

$$(w-w_0)^2 + \left(\frac{\Delta w}{2}\right)^2 = \frac{(\Delta w)^2}{2}$$

$$(w-w_0)^2 = \frac{(\Delta w)^2}{4}$$

width $= \frac{\Delta w}{2} + \frac{\Delta w}{2}$

$$w - w_0 = \pm \frac{(\Delta w)}{2}$$

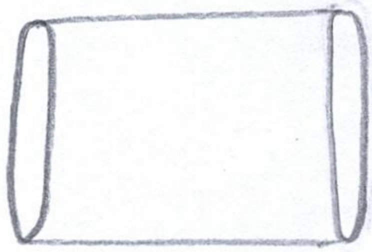
$= \Delta w$

the width at 0.5 times the amplitude

6) a) $FWHM = \Delta\omega = \frac{1}{T} = A$

$$\Rightarrow \Delta\omega = \cancel{\frac{1}{5\times10^7}} \; \underline{\underline{(5\times10^7)\,s^{-1}}}$$

b)



$\sigma_s :=$ collision cross-section

$$N_c(t) = \sigma_s \cdot \underbrace{vt}_{\text{distance traveled in time time}} \cdot \frac{N}{V} \Big] \leadsto \text{density}$$

mean free path between collisions

$$L = \frac{vt}{\sigma_s \cdot vt \cdot \frac{N}{V}} = \frac{V}{\sigma_s N}$$

$$T = \frac{V}{\sigma_s N v}$$

time between collisions

$$\cancel{P(v)\,dv_x dv_y dv_z = C \exp\left(-\frac{mv^2}{2k_BT}\right) dv}$$

$$\cancel{C \cdot \frac{\sqrt{2\pi\left(\frac{k_BT}{m}\right)}}{\sqrt{2\pi\left(\frac{k_BT}{m}\right)}} \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{v^2}{2\left(\frac{k_BT}{m}\right)}\right) dv = 1}$$

$$\cancel{\Rightarrow \; = C \cdot \sqrt{2\pi\left(\frac{k_BT}{m}\right)} = 1 \Rightarrow C = \frac{1}{\sqrt{2\pi\frac{k_BT}{m}}}}$$

$P(v) \propto$ # of particles having velocity $\vec{v}$ $\times$ # of vectors corresponding to speed v

Probability of having ~~velocity~~ $\vec{v}$ speed v

$$= C \cdot \underbrace{e^{-\frac{mv^2}{2kT}}}_{\text{Boltzmann factor}} \cdot 4\pi v^2$$

By normalization

$$C \int e^{-\frac{mv^2}{2kT}} \cdot 4\pi v^2 \, dv = 1$$

$$\left(x = v\sqrt{m/2kT} \; ; \; dx = dv \sqrt{m/2kT} \right.$$

$$4\pi C \times \left(\sqrt{\frac{2kT}{m}}\right)^3 \int e^{-x^2} \cdot x^2 \, dx = 1$$

$$\cancel{4}\pi \cdot \left(\frac{2kT}{m}\right)^{\frac{3}{2}} \cdot C \cdot \frac{\sqrt{\pi}}{\cancel{4}} = 1$$

$$\Rightarrow \boxed{C = \left(\frac{m}{2\pi kT}\right)^{3/2}}$$

$$\langle v \rangle = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot 4\pi \int_0^\infty v^3 e^{\frac{-mv^2}{2kT}} dv$$

$$= \sqrt{\frac{8kT}{\pi m}}$$

$$\tau = \frac{V}{\sigma_s N \cdot \sqrt{\frac{8kT}{\pi m}}}$$

$$= \left(\frac{V}{N}\right) \cdot \frac{1}{\sigma_s} \cdot \sqrt{\frac{\pi m}{8kT}}$$

$$PV = NkT$$

$$\frac{N}{V} = \frac{P}{kT}$$

$$\Rightarrow \tau = \frac{kT}{P} \cdot \frac{1}{\sigma_s} \cdot \frac{\sqrt{\pi m}}{\sqrt{8kT}}$$

$$\tau = \frac{\sqrt{\pi m kT}}{\sqrt{8}} \cdot \frac{1}{\sigma_s P}$$

$$\Rightarrow \boxed{\Delta W = (\sigma_s P) \cdot \left(\frac{8}{\pi m kT}\right)^{1/2}}$$

$$\tau_{collision} = \frac{1}{\sigma_s P} \cdot \left(\frac{\pi m k T}{8} \right)^{1/2}$$

$P \uparrow \quad \tau \downarrow \quad \Delta w \uparrow \Rightarrow$ low pressures are needed to reduce the effects of collisional broadening.

c) ① $P = 1 \text{ atm} = 1.01325 \times 10^5 \text{ N/m}^2$

$\sigma_s = \pi r^2 = \pi (0.17 \times 10^{-9})^2 = 9.079 \times 10^{-20} \text{ m}^2$

$T = 473.15 \text{K} \qquad m = 3.32 \times 10^{-25} \text{ kg}$

$k = 1.38 \times 10^{-23} \text{ J/K}$

$$\boxed{\Delta w = 3.15 \times 10^8 \text{ s}^{-1}}$$

✓

② $P = 10^{-3} \text{ atm}$

$$\boxed{\Delta w = 3.15 \times 10^5 \text{ s}^{-1}}$$

✓ lower pressure, sharper line

d) $$5 \times 10^7 = (9.079 \times 10^{-20}) P \cdot \left(\frac{8}{(\pi)(3.32 \times 10^{-25}) \times 1.38 \times 10^{-23} \times 473.15} \right)^{1/2}$$

$\Rightarrow P \approx 16060.3 \text{ Pa} \approx \underline{0.159 \text{ atm}}$

At Pressures slightly smaller than 0.16 atm, the pressure broadened linewidth would be a tad smaller.

e) $$g(\omega) = \frac{\Delta\omega}{2\pi} \cdot \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\Delta\omega}{2}\right)^2}$$

$$\omega = \frac{2\pi c}{\lambda} \quad \text{and} \quad \omega_0 = \frac{2\pi c}{\lambda_0}$$

$$g(\lambda) = \frac{\Delta\omega}{2\pi} \cdot \frac{1}{(2\pi c)^2\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)^2 + \left(\frac{\Delta\omega}{2}\right)^2}$$

Plug in $\frac{1}{\pi\Delta\omega}$ (half the max amplitude)

$$\frac{1}{\pi\Delta\omega} = \frac{\Delta\omega}{2\pi} \cdot \frac{1}{4\pi^2c^2\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)^2 + \left(\frac{\Delta\omega}{2}\right)^2}$$

$$4\pi^2c^2\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)^2 + \left(\frac{\Delta\omega}{2}\right)^2 = \frac{(\Delta\omega)^2}{2}$$

$$4\pi^2c^2\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)^2 = \frac{\Delta\omega^2}{4}$$

$$\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)^2 = \frac{\Delta\omega^2}{16\pi^2c^2}$$

$$\frac{1}{\lambda} - \frac{1}{\lambda_0} = \pm \frac{\Delta\omega}{4\pi c}$$

$$\frac{1}{\lambda_+} - \frac{1}{\lambda_0} = \frac{\Delta\omega}{4\pi c}$$

$$\frac{1}{\lambda_+} = \frac{\Delta\omega}{4\pi c} + \frac{1}{\lambda_0}$$

$$(\lambda_+)^{-1} = \frac{\Delta\omega\lambda_0 + 4\pi c}{4\pi c\lambda_0}$$

$$\lambda_+ = \frac{4\pi c\lambda_0}{\Delta\omega\lambda_0 + 4\pi c}$$

$$\frac{1}{\lambda_-} - \frac{1}{\lambda_0} = -\frac{\Delta\omega}{4\pi c}$$

$$\frac{1}{\lambda_-} = \frac{1}{\lambda_0} - \frac{\Delta\omega}{4\pi c}$$

$$\frac{1}{\lambda_-} = \frac{4\pi c - \Delta\omega\lambda_0}{4\pi c\lambda_0}$$

$$\lambda_- = \frac{4\pi c\lambda_0}{4\pi c - \Delta\omega\lambda_0}$$

$$\Delta\lambda = (4\pi c\lambda_0)\left(\frac{1}{\Delta\omega\lambda_0 + 4\pi c} + \frac{1}{4\pi c - \Delta\omega\lambda_0}\right)$$

$$= (4\pi c\lambda_0)\left(\frac{\cancel{2\Delta\omega\lambda_0}\ 8\pi c}{16\pi^2c^2 - \Delta\omega^2\lambda_0^2}\right)$$

$$\boxed{\Delta\lambda = \frac{32\pi^2c^2\lambda_0}{16\pi^2c^2 - \Delta\omega^2\lambda_0^2}}$$

$$16\pi^2c^2 - \Delta\omega^2\lambda_0^2 = \frac{32\pi^2c^2\lambda_0}{\Delta\lambda}$$

$$\boxed{\Delta\omega = \sqrt{\frac{16\pi^2c^2}{\lambda_0^2} - \frac{32\pi^2c^2}{(\Delta\lambda)(\lambda_0)}}}$$

7) a) $\frac{d^2\vec{r}}{dt^2} + \gamma \frac{d\vec{r}}{dt} + \omega_0^2 \vec{r} = -\frac{e\vec{E}_0}{m} e^{-i\omega t}$

$\vec{r}(t) = \vec{r}_0 e^{-i\omega t} \quad \frac{d\vec{r}}{dt} = -i\omega \vec{r}_0 e^{-i\omega t}$

$\frac{d^2\vec{r}}{dt^2} = -\omega^2 \vec{r}_0 e^{-i\omega t}$

Now,

$-\omega^2 \vec{r}_0 e^{-i\omega t} + (\gamma)(-i\omega)\vec{r}_0 e^{-i\omega t} + \omega_0^2 \vec{r}_0 e^{-i\omega t} = -\frac{e\vec{E}_0}{m} e^{-i\omega t}$

$\Rightarrow \boxed{\vec{r}_0 = \frac{-e/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \vec{E}_0}$

b) $\vec{p} = -e\vec{r} = \boxed{\frac{e^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \vec{E}_0 e^{-i\omega t}}$

c) $\vec{P} = N\vec{p} = \boxed{\left(\frac{Ne^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}\right) \vec{E}_0 e^{-i\omega t}}$

d) $(\varepsilon_r - 1)\varepsilon_0 \vec{E}_0 = \left(\frac{Ne^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}\right) \vec{E}_0$

$$\Rightarrow \boxed{\varepsilon_r = 1 + \frac{Ne^2/m}{\varepsilon_0(\omega_0^2 - \omega^2 - i\gamma\omega)}}$$

$$n = \sqrt{\varepsilon_r} = \sqrt{1 + \frac{Ne^2/m}{\varepsilon_0(\omega_0^2 - \omega^2 - i\gamma\omega)}} \in \mathbb{C}$$

e) Real part governs the phase velocity of light.
Imaginary part accounts for absorption in the medium.

f) $\omega \approx \omega_0$

$$n \approx \sqrt{1 + \frac{Ne^2}{m\varepsilon_0 \cdot -i\gamma\omega}} = \sqrt{1 + \frac{Ne^2}{m\varepsilon_0\gamma\omega} i}$$

$$\varepsilon_r = 1 + \frac{Ne^2/m}{\varepsilon_0(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

$$\left(Q := \frac{Ne^2}{m\varepsilon_0} \right.$$

$$\varepsilon_r = 1 + \frac{Q}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

$$\varepsilon_r(\omega) = 1 + \frac{Q}{\omega_0^2 - \omega^2 - i\gamma\omega} \cdot \frac{\omega_0^2 - \omega^2 + i\gamma\omega}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

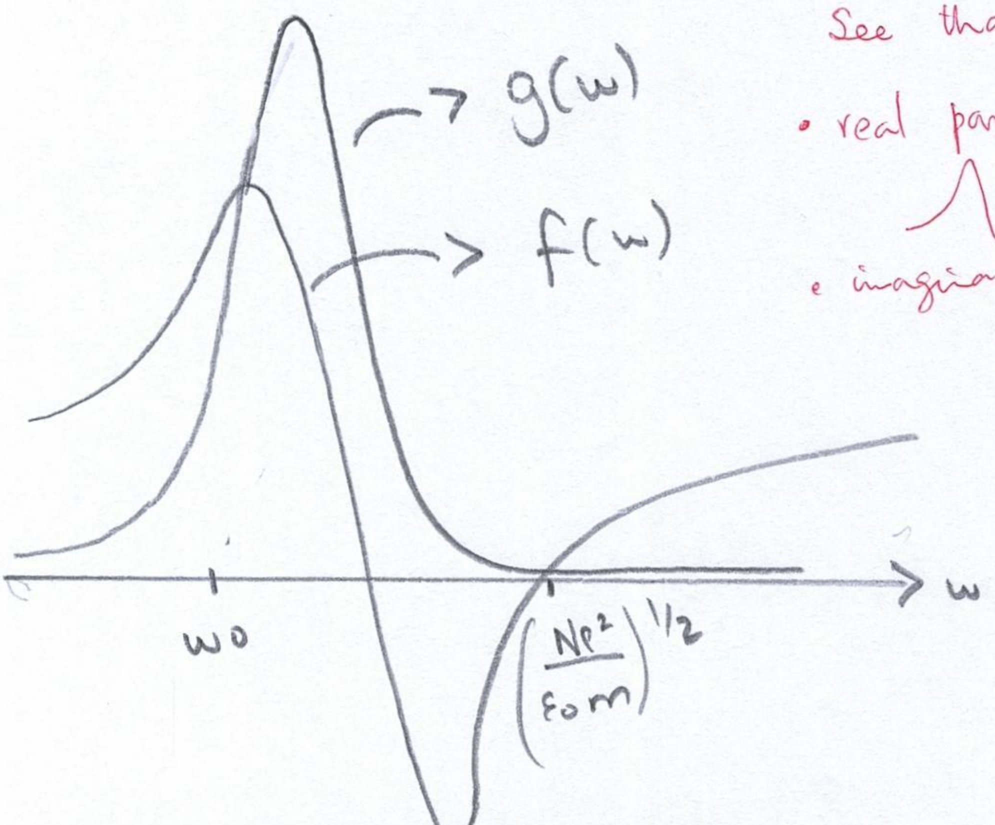
$$\varepsilon_r(\omega) = 1 + \frac{(Q)(\omega_0^2 - \omega^2 + i\gamma\omega)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$f(\omega) := \mathrm{Re}(\varepsilon_r(\omega))\ A/1 - 1$$

$$= \frac{(Ne^2/\varepsilon_0 m) \cdot (\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \checkmark$$

$$g(\omega) := \mathrm{Im}(\varepsilon_r(\omega))$$

$$= \frac{(Ne^2/\varepsilon_0 m)(\gamma\omega)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \checkmark$$



See that:

- real part is ~~absorption~~ dispersive Lorentzian
- imaginary part is absorption Lorentzian