HW 2 SOLN. Sabreh Anwar

Q1.

$$\tilde{\rho} = \begin{pmatrix} \frac{1}{2}+c & a+ib \\ a-ib & \frac{1}{2}-c \end{pmatrix} = \begin{pmatrix} \tilde{\rho}_{aa} & \tilde{\rho}_{ab} \\ \tilde{\rho}_{ba} & \tilde{\rho}_{bb} \end{pmatrix}$$

From ①

Now $u = \tilde{\rho}_{ab} + \tilde{\rho}_{ba} = 2a \Rightarrow \boxed{a = \frac{1}{2}u}$

$$v = \frac{\tilde{\rho}_{ab} - \tilde{\rho}_{ba}}{i} = \frac{1}{i}\left(a+ib-a+ib\right) = 2b \Rightarrow \boxed{b = \frac{v}{2}}$$

$$\tilde{w} = \tilde{\rho}_{bb} - \tilde{\rho}_{aa} = \frac{1}{2} - c - \frac{1}{2} - c = -2c \Rightarrow \boxed{c = -\frac{w}{2}}$$

$$\therefore \tilde{\rho} = \frac{\hat{1}}{2} + \frac{u}{2}\hat{\sigma}_x + \frac{v}{2}\hat{\sigma}_y - \frac{w}{2}\hat{\sigma}_z .$$

$$= \begin{pmatrix} \frac{1}{2} - \frac{w}{2} & \frac{u}{2} + i\frac{v}{2} \\ \frac{u}{2} - i\frac{v}{2} & \frac{1}{2} + \frac{w}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-w & u+iv \\ u-iv & 1+w \end{pmatrix} .$$

Q2 (a) On resonance

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -\begin{pmatrix} \Omega_R \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} - \gamma \begin{pmatrix} u \\ v \\ 2(w+1) \end{pmatrix}$$

no damping

$$= -\begin{pmatrix} \Omega_R \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$= -\begin{vmatrix} \hat{u} & \hat{v} & \hat{w} \\ \Omega_R & 0 & 0 \\ u & v & w \end{vmatrix} = -\left[\hat{u}(0) - \hat{v}\,\Omega_R w + \hat{w}\,\Omega_R v\right]$$

$$\frac{du}{dt} = 0; \quad \frac{dv}{dt} = \Omega_R w$$

$$\Rightarrow \frac{d}{dt}w = -\Omega_R v$$

Solve together $\frac{d^2 w}{dt^2} = -\Omega_R\left(\Omega_R w\right) = -\Omega_R^2 w$

$$\frac{d^2 w}{dt^2} + \Omega_R^2 w = 0 \xrightarrow{\text{Solve}} w(t) = A e^{+i\Omega_R t} + B e^{-i\Omega_R t} .$$

At $t=0$, $w(0) = -1$; $\boxed{-1 = A+B \rightarrow B = -(1+A)}$ — ①

Hi
Up

Now $\frac{dv}{dt} = \Omega_R w$

$\downarrow$ integrate

$$\frac{dv}{dt} = \Omega_R \left(A e^{i\Omega_R t} + B e^{-i\Omega_R t} \right)$$

$$v(t) = \cancel{\Omega_R} \left(\frac{A e^{i\Omega_R t}}{i\cancel{\Omega_R}} - \frac{B e^{-i\Omega_R t}}{i\cancel{\Omega_R}} \right)$$

$$v(t) = \frac{1}{i} \left(A e^{i\Omega_R t} - B e^{-i\Omega_R t} \right)$$

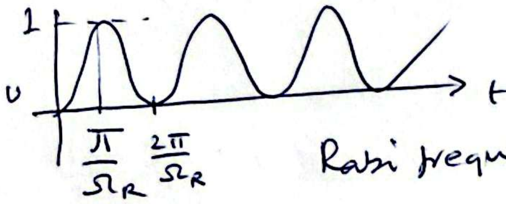
$v(0) = 0 \Rightarrow \boxed{A = B}$ — ②

From ① & ② : $A = -(1+A) \Rightarrow A = 1/2, \; B = 1/2$.

Hence $w(t) = -\frac{1}{2}\left(e^{i\Omega_R t} + e^{-i\Omega_R t}\right) = \underline{-\cos\Omega_R t}$,

$$\rho_{bb}(t) = \frac{1}{2}(1+w) = \frac{1}{2}\left(1 - \cos\Omega_R t\right) = \sin^2\left(\frac{\Omega_R t}{2}\right).$$

$$\boxed{\rho_{bb}(t) = \sin^2(\Omega_R t/2)}$$



Rabi frequency $\rightarrow \Omega_R$.

(b) $\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = -\begin{vmatrix} \hat{u} & \hat{v} & \hat{w} \\ 0 & 0 & \delta \\ u & v & w \end{vmatrix} = -\left[\hat{u}(-\delta v) - \hat{v}(-\delta u) \right]$

$\left.\begin{aligned} \frac{du}{dt} &= \delta v \\ \text{and } \frac{dv}{dt} &= -\delta u \end{aligned}\right\} \Rightarrow \frac{d^2 u}{dt^2} = -\delta^2 u \Rightarrow \underline{u(t) = A e^{i\delta t} + B e^{-i\delta t}}.$

$$\frac{du}{dt} = i\delta A e^{i\delta t} - i\delta B e^{-i\delta t}$$

$$\underline{v = iAe^{i\delta t} - iBe^{-i\delta t}}$$

At $t = 0$

$\cancel{u = 0 = A + B \Rightarrow B = -A}$

$\cancel{u(t) = 2A\cos\delta t} \;\; \cancel{2iA\sin\delta t}$

$\cancel{v(t) = 2iA\cos\delta t}$

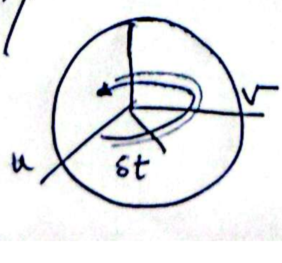
$v = 0, \quad A = B$

$$v(t) = iA\left(e^{i\delta t} - e^{-i\delta t}\right)$$

$$\boxed{v(t) = -2A\sin\delta t}$$

$$\boxed{u(t) = 2A\cos\delta t}$$

Precession due to detuning in the uv plane, coherences oscillate!



ص٣

Q3.

$$\frac{d}{dt}(\rho_{ab}) = \frac{d}{dt}(c_a c_b^*) = \frac{dc_a}{dt} c_b^* + c_a \frac{dc_b^*}{dt}$$

$$= \cancel{\frac{i\Omega_R}{2} c_b c_a^*} \quad \frac{i\Omega_R}{2}|c_b|^2 - \frac{i\Omega_R}{2}|c_a|^2$$

$$\boxed{\frac{d}{dt}(\rho_{ab}) = \frac{i\Omega_R}{2}(\rho_{bb} - \rho_{aa}) = \frac{i\Omega_R}{2}(2\rho_{bb} - 1).}$$

— we start with this. We need ρ_{bb}.

Now $\cancel{\frac{d^2 c_b}{dt} = \frac{i\Omega_R}{2}\left(\frac{i\Omega_R}{2}\right)|c_b(t)|^2}$

$\cancel{= -\frac{\Omega_R^2}{4}\rho_{bb}}$

Differentiate

$$\boxed{\frac{d^2}{dt^2}\rho_{ab} = i\Omega_R \frac{d\rho_{bb}}{dt}.}$$

— Solve this. Find $\frac{d\rho_{bb}}{dt}$.

Now $\cancel{\frac{dc_b}{dt} = \frac{i\Omega_R}{2} c_a(t)}$ Insert here

$\cancel{\frac{d^2 c_b}{dt^2} = \frac{i\Omega_R}{4} \left(\frac{i\Omega_R}{2}\right) c_b(t)}$

$$\frac{dc_b(t)}{dt} = \frac{i\Omega_R}{2} c_a(t)$$

$$\frac{d^2 c_b(t)}{dt^2} = \frac{i\Omega_R}{2}\left(\frac{dc_a}{dt}\right) = \frac{i\Omega_R}{2}\left(\frac{i\Omega_R}{2}\right) c_b = -\frac{\Omega_R^2}{4} c_b.$$

$$\Rightarrow c_b = A e^{\frac{i\Omega_R t}{2}} + B e^{-i\Omega_R t/2}$$

At $\underbrace{t=0,\ c_b = 0}_{\text{initially}}$ $\Rightarrow c_b = 2iA\sin(\Omega_R t/2) = \sin(\Omega_R t/2)$

$$\rho_{bb} = c_b^* c_b = \sin^2(\Omega_R t/2).$$

at $t = \frac{\pi}{\Omega_R}, c_b = 1 \Rightarrow A = \frac{1}{2i}$

$$\frac{d\rho_{bb}}{dt} = \frac{\Omega_R}{2}\sin(\Omega_R t)$$

So the Eq. in $\frac{d^2}{dt^2}\rho_{ab}$ becomes

$$\frac{d^2}{dt^2}\rho_{ab} = \frac{i\Omega_R^2}{2}\sin(\Omega_R t)$$

→ Assume a particular integral as solut $\rho_{ab} = F\cos(\Omega_R t) + G\sin(\Omega_R t)$

$$\frac{d\rho_{ab}}{dt} = -\Omega_R F \sin(\Omega_R t) + \Omega_R G\cos(\Omega_R t)$$

$$\frac{d^2\rho_{ab}}{dt^2} = -\Omega_R^2 F\cos(\Omega_R t) - \Omega_R^2 G \sin(\Omega_R t)$$

Compare, $F = 0$, $-\cancel{\Omega_R^2} G = \frac{i\cancel{\Omega_R^2}}{2}$

$G = -\frac{i}{2}$

$$\therefore \boxed{\rho_{ab} = -\frac{i}{2}\sin(\Omega_R t)} \quad \text{QED.}$$

There are alternative ways to solve this too, e.g. find $(c_a c_b^*)(t)$.

(a) $\hat{\mu}_{a,b} = -e\langle 2s | z | 3p_z\rangle$

$$= -e\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} \underbrace{dr\,d\theta\,d\phi\; r^2 \cancel{\cos\theta}\sin\theta}_{\text{volume metric for integratn}} \frac{\sqrt{2}}{81\sqrt{\pi}}\frac{1}{a_0^{5/2}}\frac{1}{4\sqrt{2\pi}\,a_0^{3/2}}$$

$$\times\left(6 - r/a_0\right)\left(2 - \frac{r}{a_0}\right) r\cos\theta \; \boxed{\overset{\text{from } z}{(r\cos\theta)}}$$

$$\times\, e^{-r/3a_0}e^{-r/2a_0}$$

$= 1.50\times10^{-29}$ Cm. (If I had x polarizat $\hat{z} \to r\sin\theta\cos\phi$, $\mu = 0$). These are called select rules.

(b) $B = \frac{|\mu_{a,b}|^2\pi}{3\hbar^2\varepsilon_0} \sim 2.4\times10^{21}\ \underline{m^3 J^{-1} s^{-1}}$ } Don't forget the units

$A = \frac{\hbar\omega^3}{\pi^2 c^3} B \sim 2.2\times10^{7}\ \underline{s^{-1}}$

set

Find ω $\hbar\omega = -13.6\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$ with $n_f = 3$, $n_i = 2$.

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(c) $\underbrace{\gamma = A}$ $= 2.2 \times 10^{7}\ s^{-1}$

set damping rate equal to spontaneous emission rate.

$\Omega_R = A = 2.2 \times 10^{7}\ s^{-1}$

$\frac{\Omega_R}{\gamma} = 1.$

$\Omega_R = \frac{\mu_{a,b} E_0}{\hbar}$, $E_0 = \left|\frac{\hbar \Omega_R}{\mu_{a,b}}\right| = 156\ V/m$

$I = \frac{1}{2} n c \varepsilon_0 |E_0|^2 = 32.4\ W/m^2.$

this is 1 (vacuum)

(d) $\Omega_R(t_{\pi/2}) = \pi/2 \Rightarrow \underline{t_{\pi/2} \sim 71\ ns.}$

$\frac{1}{\gamma} = \frac{1}{2.2 \times 10^{7}} = 45\ ns.$

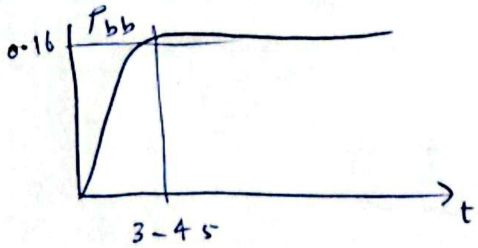
So observing Rabi oscillation will be hard (intensity is not big enough). (Damping time smaller than the 90° pulse time!).

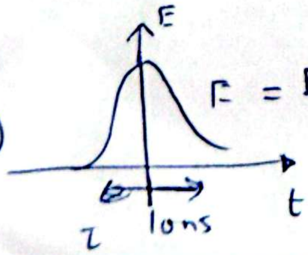
(e) when $\Omega_R = \gamma$, $S_0 = \frac{1}{2}\left(\frac{\Omega_R}{\gamma}\right)^2 = 1/2$ ... on resonance saturation.

$w_0 = -\frac{1}{1 + \frac{S_0}{1+\left(\frac{\delta\omega}{\gamma}\right)^2}} = -\frac{1}{1+S_0} = -\frac{2}{3}$

$\rho_{bb}(\infty) = \frac{w_\infty + 1}{2} = \frac{1}{6} \simeq 0.166.$

(f) Yes — see mathematica script



(g) 

$E = E_0 e^{-t^2/2\tau^2}$

We want $\int dt\ \Omega_R = \pi/2$

But $\Omega_R = \mu_{a,b} E_0(t)/\hbar$

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$$\Omega_R = \frac{\mu_{a,b} E_0}{\hbar}$$

So $$\int_{-\infty}^{\infty} dt \; \frac{\mu_{a,b} E_0}{\hbar} \, e^{-t^2/2\tau^2} = \pi/2 \qquad \text{where } \tau = 10 \text{ ns}.$$

$$\Rightarrow \quad E_0 = \frac{\hbar \, \pi/2}{\mu_{a,b} \left(\int_{-\infty}^{\infty} dt \, e^{-t^2/2\tau^2} \right)} = \left(\frac{\hbar \, \pi/2}{\underbrace{\mu_{a,b}} \sqrt{2\pi} \, \tau} \right)$$

$$E_0 = \frac{6.63 \times 10^{-34} \times \pi/2}{2\pi \times 1.5 \times 10^{-29} \times \sqrt{2\pi} \times 10 \times 10^{-9}}$$

$$= \frac{10.4 \times 10^{-34}}{236.2 \times 10^{-38}} = 0.0440 \times 10^{4} \approx 440 \quad V/m$$

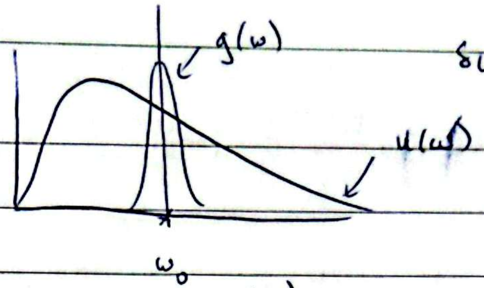
$$I_{peak} = \frac{1}{2} c \varepsilon_0 |E_{0,peak}|^2 = \frac{1}{2} \times 3 \times 10^{8} \times (440)^2 \times 8.854 \times 10^{-12}$$

$$\approx 257 \; W/m^2$$

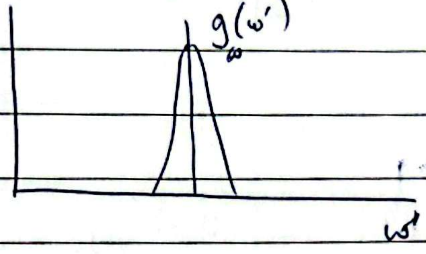
$\approx$ Three times higher electric field (peak) and $\approx$ nine times higher peak intensity is required.

~~Note on selection rules~~

Q9.6 Fox QO.



$\delta\omega \ll \Delta\omega$



$$R_{12}(\omega')\,d\omega' = N_1 B_{12}^{\omega}\, u(\omega')\, g_{\omega}(\omega')\, d\omega'$$

$$R_{21}(\omega')\,d\omega' = N_2 B_{21}^{\omega}\, u(\omega')\, g_{\omega}(\omega')\, d\omega'$$

(a) $g_{\omega}(\omega') = \delta(\omega' - \omega_0)$ $\qquad N_1 B\, u(\omega')\, d\omega'$

$$R_{12}(\omega')\,d\omega' = N_1 B_{12}^{\omega}\, u(\omega')\, \delta(\omega' - \omega_0)\, d\omega'$$

$$= N_1 B_{12}^{\omega}\, u(\omega_0)\, d\omega'$$

$$R_{12}^{total} = \int_{\omega'=0}^{\infty} R_{12}(\omega')\,d\omega' = \text{rate of absorption}$$

$$= N_1 B_{12}^{\omega}\, u(\omega_0).$$

$$R_{21}^{total} = N_2 B_{21}\, u(\omega_0) = \text{rate of emission}$$

(b) Consider a sharp laser beam

$$u(\omega') = u_{\omega}\, \delta(\omega' - \omega)$$

$$R_{12} = N_1 \int d\omega'\, B_{12}^{\omega}\, u_{\omega}\, \delta(\omega' - \omega)\, g_{\omega}(\omega') = N_1 u_{\omega} B_{12}^{\omega} g_{\omega}(\omega)$$

$$R_{21} = N_2 u_\omega B^\omega_{21} g_\omega(\omega)$$

$$\frac{dN_2}{dt} = R_{12} - R_{21} = N_1 u_\omega B^\omega_{12} g_\omega(\omega) - N_2 u_\omega B^\omega_{21} g_\omega(\omega) - N_2 A$$

$$\frac{dN_2}{dt} = u_\omega B^\omega g_\omega(\omega) (N_1 - N_2) - N_2 A$$

$$\frac{dN_2}{dt} = u_\omega \underbrace{\boxed{B^\omega g_\omega(\omega)}}_{B'} N_1 - (u_\omega \underbrace{\boxed{B^\omega g_\omega(\omega)}}_{B'} + A) N_2$$

$$dN_2/dt = u_\omega B' N_1 - (u_\omega B' + A) N_2$$

$$dN_2/dt + (u_\omega B' + A) N_2 = u_\omega B' N_1$$

$$e^{(u_\omega B' + A)t} \frac{dN_2}{dt} = u_\omega B' N_1 e^{(u_\omega B' + A)t}$$

$$+ (u_\omega B' + A) N_2 e^{(u_\omega B' + A)t} = u_\omega B' N_1 e^{(u_\omega B' + A)t}$$

$$\frac{d}{dt}\left(N_2 e^{(u_\omega B' + A)t}\right) = u_\omega B' N_1 e^{(u_\omega B' + A)t}$$

$$N_2 e^{(u_\omega B' + A)t} = \frac{u_\omega B' N_1 e^{(u_\omega B' + A)t}}{(u_\omega B' + A)} + K$$

$$N_2 = \frac{u_\omega B' N_1}{u_\omega B' + A} + K e^{-(u_\omega B' + A)t}$$

$$N_2 = \frac{u_\omega B' N_1}{u_\omega B' + A}\left(1 - e^{-(u_\omega B' + A)t}\right)$$

$$N_1 = N - N_2.$$

$$\left(\frac{N_2}{N-N_2}\right) = \left(\frac{u_\omega B'}{u_\omega B' + A}\right)\left(1 - e^{-(u_\omega B' + A)t}\right)$$

$$\frac{dN_2}{dt} = u_\omega B'(N - N_2) - (u_\omega B' + A) N_2$$

$$= u_\omega B' N - 2u_\omega B' N_2 - A N_2$$

$$\frac{dN_2}{dt} + (2u_\omega B' + A) N_2 = u_\omega B' N$$

energy density → sharp laser line (narrow band source)

narrowband

$$\boxed{\frac{N_2}{N} = \left(\frac{u_\omega B'}{2u_\omega B' + A}\right)\left(1 - e^{-(2u_\omega B' + A)t}\right) \quad \text{where } B' = B^\omega g_\omega(\omega)}$$

→ very intensive (ignore spont. for intense fields) $\frac{N_2}{N} = 1/2$ | very weak $\frac{u_\omega B'}{A} = u_\omega \frac{B^\omega g(\omega)}{A}$.

light intensity, lineshape.

Broadband

$$\frac{dN_2}{dt} = N_1 B u(\omega) - N_2 B u(\omega) - N_2 A.$$

$$= N B u(\omega) - 2 N_2 B u(\omega) - N_2 A$$

$$\boxed{\frac{N_2}{N} = \left(\frac{u(\omega) B}{2u(\omega) B + A}\right)\left(1 - e^{-(2u(\omega)B + A)t}\right)}$$

$$\cdot\ \bar{\rho}_{bb}(\infty) = \frac{1}{2}\left(\frac{S}{1+S}\right) \approx \underset{S\ \text{small}}{\frac{S}{2}} = \underset{S_\omega \to 0}{\frac{S_0}{2}} = \frac{1}{4}\left(\frac{\Omega_R}{\gamma}\right)^2 = \left(\frac{E_0^2\ \mu_{12}^2}{4\gamma^2\ \hbar^2}\right)$$

$\boxed{u_\omega \propto E_0^2}$ $\boxed{A \propto \gamma}$

$\boxed{B \propto \mu_{12}^2}$ $\boxed{g(\omega) \propto 1/\gamma}$ → line shape contributi.